ANALYSIS
using
ultrasmall numbers

How to read a proof

The goal of this paper is to help instructors or examiners understand a proof which uses ultrasmall numbers. This can happen when, in our school, final exams have an external juror in addition to the teacher. It can also be used to see that for the trained mathematician, the translation back to "classical" methods is quite straightforward.
This part does not explain how to write the proofs, only how to read them.

As in any book about analysis, we do not give the axioms of set theory, but state instead, properties of real numbers.

Observability

Extra Axioms – called also Principles – are used which allow to make an extra distinction within the real numbers: observability.

The intuition is that "ordinary numbers" are observable but that there are extremely small numbers (ultrasmall) which are so tiny that they are not observable. But if one zooms in to observe these tiny numbers, one can still see the previously observable numbers. And if such a tiny number h is added to 2, then $2+h$ is less observable than 2, which remains observable when $2+h$ is observable.

Given x and y, then x is as observable as y, or y is as observable as x. They may have the same observability. Observability is transitive.

One can consider the metaphor of scales of observation.

Numbers defined without the concept of observability are observable relative to any real number: they are always observable – or standard.

Ultrasmall

Relative to any real number, there exist ultrasmall numbers: numbers which are less, in absolute value, than any strictly positive observable number, yet not zero.

Relative to a, if h is ultrasmall, then $\frac{1}{h}$ is ultralarge. But then we also have that a and $a+h$ are extremely close, written $a \simeq a+h$, where the only new symbol is "$\simeq$" which reads "ultraclose": a difference which is ultrasmall or zero. Note that $a+h$ is not as observable as a.

An ultrasmall number has the "flavour" of an infinitesimal.

Context

Given a formula which has some parameters, the concept of ultrasmall, ultralarge or ultraclose always refer to the whole list of parameters. If h is ultrasmall, it must be ultrasmall relative to each parameter. Thus "$\simeq$" by its definition refers to all of the parameters.

The context of a statement is the list of parameters used in that statement. Some instructors use the word "observability" instead of context. This is a pedagogical choice which does not change the theory.

Closure

Non observable numbers do not show up as results of operations if they are not introduced explicitly. Given f and a, then $f(a)$ is observable (the context being a and the parameters of f).

Observable neighbour

For any number x which is not ultralarge, there is an observable number a such that $x \simeq a$.

Or:

Any number x which is not ultralarge can has be written in the form $x = a+h$ where a is observable and h is ultraclose to zero. Then a is the observable neighbour of x.

The existence of the observable neighbour (or observable part) is equivalent to the completeness of $\mathbb{R}$.

Limit

$\lim\limits_{x\to a} f(x) = L$ is, here, defined by, L is observable and

$$x \simeq a \ (x \neq a) \ \Rightarrow f(x) \simeq L$$

and does not depend on the choice of x.

Since the sum of two ultrasmall numbers is ultrasmall or zero and similar algebraic properties, it is easy to prove that the sum, resp. product, quotient of limits is the limit of the sum, resp. product, quotient.

Translation

When reading dx or $h \simeq 0$, consider that h is a real number with the idea of it being arbitrarily small in absolute value. When we keep only the observable part of the result (if it exists), this translates to "limit when h tends to 0".

Examples

Continuity

f is continuous at a if $f(a + dx) \simeq f(a)$ whenever $dx \simeq 0$. This translates to $\lim_{h \to 0} f(a + h) = f(a)$ as usual. It is also written in a simple form: $x \simeq a \Rightarrow f(x) \simeq f(a)$

Derivative of f at a

If there is an observable value D such that for any dx we have $\dfrac{f(a + dx) - f(a)}{dx} \simeq D$ then $f'(a) = D$. This translates to $\lim_{h \to 0} \dfrac{f(a + h) - f(a)}{h} = D$.

(The context is given by the parameters of f and a)

Example proof: Continuity of $f \circ g$

Assume that g is continuous at a and f is continuous at $g(a)$. Then if $x \simeq a$ we have $g(x) \simeq g(a)$ by continuity of g and $f(g(x)) \simeq f(g(a))$ by continuity of f.