Sequences and Series

This version has proofs and comments for the teacher

These come in frames like this one, and for this reason, the page numbers are not the same as on the student handout version.

The proofs given must not be understood as “the” proofs, but as the ones which over the years, I feel most comfortable with. When a theorem does not need anything specific to ultracalculus, the proof is omitted.

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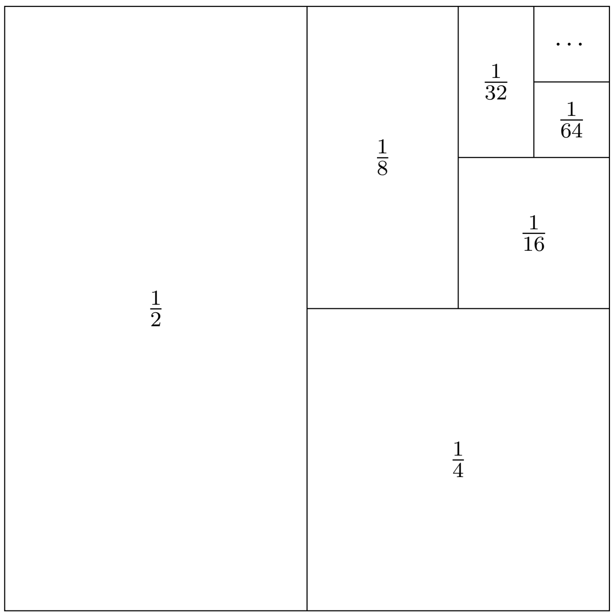
1 Infinite sums

The question is: can an unending sum give a result? Does

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

have a meaning?



Exercise 1

Is it correct to write

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \cdots = \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots \right) = \frac{1}{2} + \frac{1}{2} \cdot x$$

hence $x = \frac{1}{2} + \frac{1}{2}x$ and therefore $x = 1$?

Exercise 2

Using the same method as above, calculate:

$$x = 1 + 2 + 4 + 8 + 16 + \ldots$$

The question is: what went wrong? In order to answer this sort of question, we will first study another type of unending process.

This exercise is not a joke! The method of fiddling around to find the fraction equal to, say, $0.\overline{28}$ as seen in middle school does exactly this and is terribly wrong. Closing a parenthesis *after* infinitely many numbers is impossible unless we define what we mean. And the only definition I know of is that a series is equal to its limits *if the limit exists!* It is possible to define the closing parenthesis as "the limit if it exists" but the whole point of this chapter is to make students aware of the whole concept. So I start by assuming that nothing works until defined and proven...

2 Sequences

Definition 1
*A **sequence** is a function*

$$u : \{k, k+1, \dots\} \subseteq \mathbb{N} \longrightarrow \mathbb{R}, \quad \textit{with } k \in \mathbb{N}.$$

We also use the notation:

$$(u_n)_{n \geq k}$$

for the sequence above, with $u_n = u(n)$, for $n \geq 0$. We also write (u_n) if the set of indices is obvious or irrelevant. The numbers u_n are called the **terms** of the sequence.

The context of a sequence is the list of parameters used in its definition, in particular it contains the integer k – but not n which is a variable.

A finite sequence can be given by enumeration:

$$1, 2, 3, 4, 5$$

If the rule for obtaining the elements is obvious, dots will be used:

$$1, 2, 3, \dots, 20, 21, 22$$

If the sequence is infinite, the enumeration is impossible, but if the rule is obvious, dots will be used to indicate the never ending succession:

$$1, 2, 3, 4, 5, \dots$$

Sets are noted with braces {...} If the general term of the sequence can be represented by a_n the notation $\{a_n\}_n$ is be used.

The first and last elements of a sequence will be indicated by superscripts and subscripts:

$$(a_n)_{n=1}^{20}$$

is the notation for the sequence $a_1, a_2 \dots a_{19}, a_{20}$

Exercise 3
Find $(a_n)_n$ for the sequence of all natural numbers

For non ending sequences, the symbol $(a_n)_1^\infty$ can be used to indicate that its number of terms exceeds all finite terms. ([1])

Definition 2 (Explicit Relation)
An explicit relation expresses the kth term as a function of k.

[1] The ∞ sign expresses the idea of an unending process. This symbol does not represent a number, not even an ultralarge number. It means "never ending"

Exercise 4

Write the first terms of the following sequence:

$$\left(\frac{1}{n}\right)_1^\infty$$

Exercise 5

Write down the first five terms of the sequences specified by their nth terms (in each case, $n \in \mathbb{N}$)

(1) $u_n = 4n$

(2) $t_n = 2^{n-1}$

(3) $a_n = 3n - 2$

(4) $b_n = 2n^2 - 1$

(5) $r_n = (-1)^n$

(6) $e_n = (-1)^n \dfrac{n^2}{n+1}$

Definition 3 (Convergence)

*Let $(u_n)_{n\geq k}$ be a sequence. We say that $(u_n)_{n\geq k}$ **converges** if $\lim\limits_{n\to\infty} u_n$ exists i.e., if there is an $L \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} u_n = L.$$

or (explicitly)

*$(u_n)_{n\geq k}$ **converges** if there is an observable $L \in \mathbb{R}$ such that for all ultralarge N*

$$u_N \simeq L.$$

*The number L is the **limit** of the sequence.*

Definition 4 (Non convergence)

*A **non-convergent** sequence may be*

- ***divergent** (the terms eventually get ultralarge),*
- ***bounded***

! If a sequence has a limit, it does not necessarily "reach" its limit. It may or may not have terms equal to its limit.

Exercise 6

(1) Find the limit of $\left(\frac{1}{n}\right)_n$

(2) Find the limit of $\left(\frac{n}{n+1}\right)_n$

Definition 5 (Recurrence Relation)
A ***recurrence relation*** *expresses the kth element of a sequence in terms of one or more of its predecessors.*
In order to know where the sequence begins, it is necessary to state the value of the first term of the sequence. [a]

[a] Recall the conditions for an induction proof.

Exercise 7

Write the first terms of the sequences:

(1) $u_1 = 1 \quad u_k = \frac{u_{k-1}}{k+1}$

(2) $u_1 = 1 \quad u_n = 2 \cdot u_{n-1}$

(3) $u_1 = 2 \quad u_i = 3 + 2u_{i-1}$

If possible, rewrite them in explicit form. Why do you need an induction proof?

Exercise 8

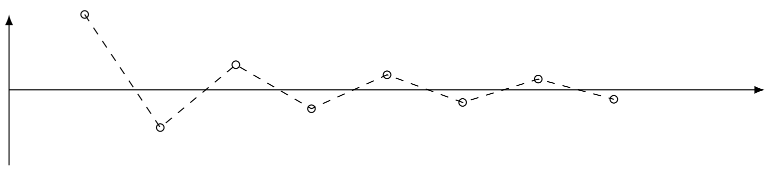
Consider
$u_1 = 5$

$$u_n = \begin{cases} \frac{u_{n-1}}{2} & \text{if } u_{n-1} \text{ is even} \\ 3 \cdot u_{n-1} + 1 & \text{if } u_{n-1} \text{ is odd} \end{cases}$$

Use other values for u_1 and try to see the behaviour of this strange sequence. (The fact that it ends in the same way for any initial value is the Syracusa conjecture.)

A **Graph** of a sequence helps to see how it behaves. Joining the plotted dots by a dotted line (because the domain is defined on natural numbers, the plot will be discrete points)

The sequence $\{1, -1/2, 1/3, -1/4, 1/5, -1/6, 1/7, -1/8, \}$ is plotted below:



Exercise 9

Give the first terms of the following sequences:

(1) $u_1 = 5 \quad u_n = 1 + \frac{u_{n-1}}{10}$

(2) $u_1 = 0 \quad u_n = \frac{1}{5 - u_{n-1}}$

If possible, rewrite them in explicit form.

Exercise 10

Write the first terms of the following sequences:

(1) $u_1 = 0 \quad u_2 = 1 \quad u_r = 2u_{r-1} - u_{r-2}$

(2) $u_1 = 1 \quad u_2 = 3 \quad u_k = 3u_{k-1} - 2u_{k-2}$

Exercise 11

One of the most famous sequences: The Fibonacci sequence. [2]

$$u_1 = 0 \quad u_2 = 1 \quad u_n = u_{n-1} + u_{n-2}$$

(1) Write the first terms (at least ten) of the sequence and describe the behaviour of the sequence.

(2) Make a new sequence v_k with the following rule (with u_n the sequence just calculated), sketch the first terms and describe the behaviour.
$v_1 = \dfrac{u_2}{u_1} \quad v_2 = \dfrac{u_3}{u_2} \quad v_3 = \dfrac{u_4}{u_3} \quad v_n = \dfrac{u_{n+1}}{u_n}$

(3) Use the same rule as in the beginning, but start with any two numbers for u_1 and u_2 (even with $u_2 > u_1$) and calculate the second sequence v_n made from these terms and sketch the first terms. Describe the behaviour.

(4) Write the first terms of the sequence: $w_1 = 1 \quad w_n = 1 + \frac{1}{w_{n-1}}$ Describe the behaviour of the sequence.

Example: Let a and d be two real numbers and let k be a positive integer. We define an **arithmetic progression** (with **common difference** d) as follows:

$$u_k = a \text{ and } u_{n+1} = u_n + d \text{ for } n \geq k.$$

It is immediate that $u_n = a + (n-k) \cdot d$, for all $n > k$. The context of this sequence is given by a, d, k.

Example: In a similar way, given $a, r \in \mathbb{R}$ and $k \in \mathbb{N}$, we define a **geometric progression** (with **common ratio** r) by

$$u_k = a \text{ and } u_{n+1} = u_n \cdot r \text{ for all } n > k.$$

Then $u_n = a \cdot r^{n-k}$ for all $n < k$. A context of this sequence is given by a, r, k.

Exercise 12

What are the conditions for an arithmetic sequence to converge?
What are the conditions for a geometric sequence to converge?

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[2] Fibonacci was an Italian mathematician in the XIIth century; it was he who introduced Arab-Indian numerals into Europe.

Exercise 13

Describe the behaviour of the following sequences:

(1) $((-1)^n)_n$

(2) $u_1 = 1 \quad u_{n+1} = 1 - \dfrac{1}{1+u_n}$

(3) $\left(\cos\left(n\frac{\pi}{3}\right)\right)_n$

(4) (random numbers between -1 and 1)

(5) $u_n = n(n+1)(n+2)$

(6) $(n)_n$

(7) $u_1 = 1 \quad u_2 = 1 \quad u_n = \dfrac{u_{n-1}}{u_{n-2}}$

Definition 6

The sequence $(u_n)_{n\geq k}$ is:

(1) ***increasing*** *if $u_n \leq u_m$ for all $m \geq n \geq k$,*

(2) ***decreasing*** *if $u_n \leq u_m$ for all $m \geq n \geq k$,*

(3) ***monotone*** *if $(u_n)_{n\geq k}$ is either increasing or decreasing.*

(4) ***bounded above*** *if there is an $M \in \mathbb{R}$ such that $u_n \leq M$ for all $n \geq k$ (the number M is an* ***upper bound****),*

(5) ***bounded below*** *if there is an $M \in \mathbb{R}$ such that $u_n \geq M$ for all $n \geq k$ (the number M is a* ***lower bound****),*

(6) ***bounded*** *if the sequence is either bounded above and bounded below.*

Let $(u_n)_{n\geq k}$ be a sequence. If it is bounded above then by the context principle there is an observable M which is also an upper bound. Conversely, if there is an observable M such that

$$u_n \leq M, \text{ for all observable } n,$$

then by the context principle this statement is true for all integers (including ultralarge integers). The same remark holds for lower bounds.

Definition 7 (Least Upper Bound)

A least upper bound M to a nonempty set A of real numbers is a value such that $x > M \Rightarrow x \notin A$ and for any $N < M$ there is an $x \in A$ such that $x > N$.

A similar definition holds for greatest lower bound.

Theorem 1

A nonempty set of real numbers bounded above has a least upper bound (l.u.b). A nonempty set of real number bounded below has a greatest lower bound (g.l.b)

The proof of theorem 1 needs closure in two versions: the usual one:

"If there is an x satisfying a property, then there is an observable x satisfying that property."

and its contrapositive

"If all observable x satisfy a property, then all x satisfy that property."

Exercise 14

Assume a set A has an upper bound.
The proof of theorem 1 requires to justify the following steps:

- Then there is an observable $a \in A$ and an observable upper bound B. Justify.
- Let N be ultralarge. Divide the interval $[a, B]$ in N even parts. Let $x_k = a + k \cdot \frac{B-a}{N}$. Let a_j be the smallest value in the partition which is still an upper bound for A and let c be its observable neighbour.

  Justify that x_j has an observable neighbour.
- c is the least upper bound. Explain and this ends the proof.

Reminder: this is the completeness of real numbers: a crucial property not shared by rational numbers – think of $\{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 \leq 2\}$ which has no l.u.b. in $\mathbb{Q}$.

First I mention that if a set is a closed interval, $[1, 5]$ then 5 is a l.u.b. If it is open, $]1, 5[$, then 5 is still the l.u.b. My advice; take your time on this one, it is not easy to grasp.

But a set can be bits and pieces, not intervals.
Consider a set A and an upper bound m for A. The context is given by A (i.e., the parameters used in its definition). By closure, since A is assumed not empty, there is an element in A hence there is an observable $a \in A$. Similarly, if there is an upper bound, then by closure, there is an observable upper bound b.([a])
Let N be an ultralarge whole number. Divide $[a, b]$ into N even parts as usual. Let x_j be the smallest among partition points which is still an upper bound. That means that either $x_{j_1} \in A$ or there is an $x \in A$ such that $x_{j-1} < x$.
Since x_j is between a and b it is not ultralarge and therefore has an observable neighbour, c.
Let $d \in [a, b]$ and observable, then if $d > c \simeq x_j$ then, in $d > x_j$ hence $d \notin A$. If $d < c$ then $d < x_{j-1}$ hence there is a $x \in a$ such that $d < x$. So c is the least upper bound among all observable numbers, hence it is the least upper bound among all numbers.

(If there were a counter example, there would be an observable one...)

[a]This characterisation can also be used for functions: if a function or sequence has ultralarge values, then it has no maximal value since if it did, it would be an observable maximum.

Exercise 15
This theorem is *not* true if one replaces "real" by "rational". Consider

$$\{x \in \mathbb{Q} \mid x^2 < 2\}$$

Why does this not have a a least upper bound?

Exercise 16
prove the following theorem:

Theorem 2 (Monotone Convergence)
Any increasing sequence which is bounded above is convergent and has a limit. Similarly, any decreasing sequence which bounded below is convergent and has a limit.

If the sequence is bounded above, then it has a l.u.b.
We need to show that the l.u.b. is the limit.
Let L be the l.u.b of (u_n). If for all n, $u_n \not\simeq L$, then, since the sequence is increasing, there is an observable number $L' < L$ such that for all n we have $u_n < L'$ hence L would not be the l.u.b.
So there is an n such that $u_n \simeq L$ and since for all k, we have $u_{n+k} \geq u_n$ and L is l.u.b, all elements of the sequence after n satisfy $u_{n+k} \simeq L$. And this defines L as the limit.

! A sequence (u_n) is really a function $u(n)$ hence its context depends on the parameters used to define the terms. If you take h ultrasmall relative to the standard context and the constant sequence $h, h, h, h, h, h, \ldots$ then the context of this sequence is h and its limit is h, not 0.

Students have come with these questions...

Construction of a sequence to calculate $\sqrt{2}$

The square root of 2 (or any number) can be computed by repeated approximation. Here is one of many methods:

Let $\sqrt{a}$ be a first approximation to $\sqrt{2}$ (the leftover part b is such that $a + b = 2$)
$\sqrt{a+b} = \sqrt{a} + \delta$ where δ is the error on the result
The approximation is such that we hope that $\delta < 1$, neglect δ^2 which is even smaller, thus obtaining the following approximation:

$$a + b = a + 2\sqrt{a}\delta + \delta^2 \approx a + 2\sqrt{a}\delta$$

from which we get

$$\frac{b}{2\sqrt{a}} \approx \delta$$

thus $\sqrt{a}+\frac{b}{2\sqrt{a}}$ will be a better approximation than $\sqrt{a}$

Let $\sqrt{a}$ be written v, then $a=v^2$ and as $a+b=2$ we have $b=2-v^2$, thus $v+\frac{2-v^2}{2v}$ is a better approximation to $\sqrt{2}$ than v.

A sequence can be constructed:

$a_{n+1}=a_n+\dfrac{2-a_n^2}{2a_n}$ $\quad a_1=$first approximation

Compute the first terms of the sequence for different approximations: $a_1=1$, $a_1=1.5$ or even $a_1=2$

The following values are first a computer value for $\sqrt{2}$, followed by sequence value: a_8 with $(a_1=1.5)$

1.4142135623730950488016887242096980785696718753769480731766797379 8...

1.4142135623730950488016887242096980785696718753769480731766797380 0...

Exercise 17

Write the sequence that calculates $\sqrt{3}$ and calculate the first approximations.

Definition 8

Let $(u_n)_{n\geq k}$ be a sequence. We say that $(u_n)_{n\geq k}$ is a ***Cauchy sequence*** *if*

$$u_{N'}\simeq u_N, \quad \textit{for all positive ultralarge integers } N,N'.$$

A context is given by the sequence. By the context principle, a sequence is a Cauchy sequence if and only if this condition is met for any extended context.

Exercise 18

Prove the following theorem:

Theorem 3

Let $(u_n)_{n\geq k}$ be a sequence. Then (u_n) converges if and only if $(u_n)_{n\geq k}$ is a Cauchy sequence.

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(1) assume it converges to L. Then for any ultralarge N and N' we have $u_N\simeq L\simeq u_{N'}$ hence $u_N\simeq u_{N'}$.

(2) assume it is a Cauchy sequence. Then for any N, N', we have $u_N\simeq u_{N'}$, hence there is an x such that for all ultralarge M, $u_M\simeq x$. (Take $x=u_N$). Hence there is an observable c such that for all ultralarge M, $u_M\simeq c$, so c is the limit.

In particular, this shows that a Cauchy sequence does not reach ultralarge values.

Exercise 19

Back to Fibonacci. Use theorem 3 to prove that the sequence of ratios converges. Show that the recurrence relation has a fixed point. Then show that this fixed point is the limit.

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3 Series

Let $(u_n)_{n\geq k}$ be a sequence. It is possible to define another sequence by considering the **partial sums** $s_k = u_k$ and $s_{n+1} = s_n + u_{n+1}$, for $n \geq k$. In other words, for a positive integer N we have

$$s_N = u_k + u_{k+1} + \cdots + u_N = \sum_{n=k}^{N} u_n.$$

Definition 9 (Partial Sum)
*A **partial sum** is the sum up to a given index number. It is indicated by*

$$\mathcal{S}_1 = \sum_{i=1}^{1} \quad \mathcal{S}_2 = \sum_{i=1}^{2} \quad \mathcal{S}_n = \sum_{i=1}^{n}$$

Definition 10 (Infinite Series)
*An infinite series is the **limit** of its partial sums.*
*An infinite series has a sum **iff** it converges.*
*Let $(u_n)_{n\geq k}$ be a sequence. A **series** is the sequence*

$$\left(\sum_{n=k}^{N} u_n\right)_{N\geq k}$$

of the partial sums. We will denote this series by

$$\sum_{n=k}^{\infty} u_n.$$

This definition is equivalent to:

Definition 11 (Convergence of a Series)
*A series converges **iff** there is an observable L such that for any ultralarge N*

$$\sum_{i=k}^{N} u_i \simeq L$$

Partial sums represent successive approximations of the total sum

$$u_k + u_{k+1} + u_{k+2} + \ldots$$

which is not necessarily a real number in the sense that it is not guaranteed that the partial sums converge.

Definition 12
*Let $\sum_{n\geq k} u_n$ be a series. We say that $\sum_{n\geq k} u_n$ **converges** to L if the sequence of patial sums converge to L.*

If the series converges, then the total sum is equal to the limit of the sequence of partial sums. As before, if the limit exists, it is observable

Exercise 20

Here is a well known series for which it may be possible to guess the limit. However, the question is how to prove it.

Calculate

$$\sum_{i=0}^{\infty} \frac{1}{2^i}$$

Exercise 21

Same question (also graphically) for

$$\sum_{k=1}^{\infty} \frac{1}{4^k}$$

Exercise 22

Same question for

$$\sum_{k=1}^{\infty} \frac{1}{n^k}$$

Example: Consider the **arithmetic series** $\sum_{n\geq 1} u_n$, with $u_1 = a$ and $u_n = a + (n-1) \cdot d$.

A context is given by u. To establish the value of $\sum_{n=1}^{N} u_n$ first note that

$$1 + 2 + \cdots + N - 1 = \frac{1}{2}(1 + (N-1) + (2 + (N-2)) + \cdots + (N-1) + 1)) = \frac{N \cdot (N-1)}{2}$$

thus

$$\sum_{n=1}^{N} a+(n-1)\cdot d = N\cdot a+d\sum_{n=1}^{N} n-1 = N\cdot a+d\cdot\frac{N \cdot (N-1)}{2} = \frac{N}{2}(2a+(N-1)d) = \frac{N}{2}(u_1+u_N).$$

If N is ultralarge, then

$$\sum_{n=1}^{N} u_n = \frac{N}{2}(2a + (N-1)d)$$

is also ultralarge.

Hence an arithmetic series cannot converge unless $a = d = 0$.

Example: Consider the **geometric series** $\sum_{n\geq 1} u_n$, with $u_1 = a$ and $u_n = a \cdot r^{n-1}$, with $a, r \in \mathbb{R}$ ($a \neq 0$). Let $s_N = \sum_{n\geq 1}^{N} u_n$. Note that:

$$s_N = a + ar + ar^2 + \cdots + ar^{N-1}$$

multiply by $(1-r)$ and obtain $a - ar^N = a(1-r^N)$

therefore $s_N \cdot (1-r) = a \cdot (1-r^N)$ and

$$s_N = a \cdot \frac{1-r^N}{1-r}, \quad \text{if } r \neq 1.$$

Note that if $r = 1$ then $s_N = a \cdot N$ so with a common ratio equal to 1, if the initial term is not zero, the series diverges.

It is simple to check that

$$\sum_{n \geq 1} a \cdot r^n \begin{cases} \text{diverges if } |r| \geq 1 \\ \text{converges to } a \cdot \frac{1}{1-r} \text{ if } |r| < 1. \end{cases}$$

Exercise 23

For a geometric series with 2 as first term and $r = 3/4$. Write the first terms. Calculate the limit.

Exercise 24

Let

$$\sum_{k=0}^{\infty} 3 \cdot 10^{-k}$$

Does this series converge and if so, what is its limit?

Exercise 25

Calculate (if the value exists)

$$\sum_{j=0}^{\infty} 0.999^j$$

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4 Convergence Criteria

Exercise 26

Use theorem 3 to prove the following.

Theorem 4

Let $\sum_{n \geq k} u_n$ *be a series. Then* $\sum_{n \geq k} u_n$ *converges if and only if*

$$\text{for any ultralarge numbers } N < N' \quad \sum_{n=N}^{N'} u_n \simeq 0$$

The sequence of partial sums is a Cauchy sequence.

Theorem 5 (Comparison test)
Let $(u_n)_{n\geq k}$ and $(v_n)_{n\geq k}$ be two sequences with non-negative terms such that

$$u_n \geq v_n, \quad \text{for each } n \geq k.$$

If the series $\sum_{n\geq k} u_n$ converges then the series $\sum_{n\geq k} v_n$ converges also.

The series for u_n provides un upper bound for v_n: take ultralarge N, and call U the limit for u_n. Then $U \simeq u_N > v_N$.

The contrapositive of the previous theorem can be used to prove the divergence of a series.

Theorem 6
Let (u_n) be s sequence of positive terms. If $\sum_{n\geq k} u_n$ converges then, for ultralarge N $u_N \simeq 0$.

Theorem 4 with ultralarge $N-1$ and N

Example: The converse of this theorem is false: consider the **harmonic series**

$$\sum_{n\geq 1} \frac{1}{n}.$$

We have $\lim_{n\to\infty} \frac{1}{n} = 0$. We will show now that this series diverges.

We observe that

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}$$

$$s_8 = s_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geq s_4 + 4 \cdot \frac{1}{8} = s_4 + \frac{1}{2} \geq 1 + 3 \cdot \frac{1}{2}.$$

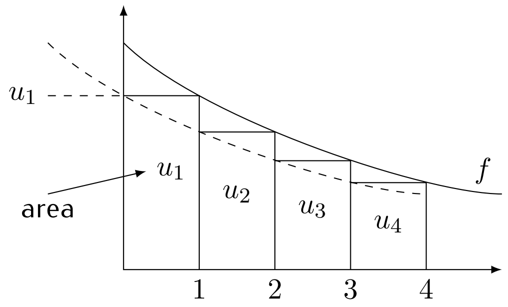
By induction, we see that

$$S_{2^N} \geq 1 + N \cdot \frac{1}{2}.$$

But this implies that the series diverges because if N is ultralarge then 2^N is ultralarge and $S_{2^N} \geq 1 + \frac{N}{2}$ is ultralarge hence not observable.

Theorem 7 (Integral Test)
Let $f : [k, \infty[\to \mathbb{R}$ be a continuous decreasing and positive function. Let $F(N) = \int_k^N f(x) \cdot dx$. Then the series $\sum_{n\geq k} f(n)$ converges if and only if $\lim_{N\to\infty} F(N)$ exists.

The following sketch show that the integral will be an upper bound for the series. The dashed line is the integral shifted to the left which is a lower bound. Hence the integral converges iff the series converges.



I do not formalise this more...

Example: Consider the series $\sum_{n\geq 1} \frac{1}{n^2}$. Let $f :]0, \infty[$ by $x \mapsto \frac{1}{x^2}$. It is a positive continuous decreasing function. Then

$$F(N) = \int_1^N \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_1^N = 1 - \frac{1}{N}$$

Then $\lim_{N\to\infty} F(N) = 1$ so the series $\sum_{n\geq 1} \frac{1}{n^2}$ converges.

Exercise 27

The **Riemann series** is

$$\sum_{n\geq 1} \frac{1}{n^p} \quad \text{with } p \in \mathbb{R}.$$

Show that the Riemann series converges if and only if $p > 1$.

The two following criteria use comparisons with some geometric series.

Theorem 8 (Ratio Test)
Let $\sum_{n\geq k} u_n$ *be a series with strictly positive terms.*
If

$$N \text{ is ultralarge } \Rightarrow \frac{u_{N+1}}{u_N} \simeq L$$

then

(1) if $L > 1$ the series diverges,

(2) if $L < 1$ the series converges.

The ratio test is inconclusive in the case $L = 1$: we have seen that $\sum_{n\geq 1} \frac{1}{n}$ diverges but $\sum_{n\geq 1} \frac{1}{n^2}$ converges.

Assume $L < 1$ then there is an observable r such that, for ultralarge N

$$\frac{u_{N+1}}{u_N} \simeq L < r < 1$$

then for all ultralrge N, we have $u_{N+1} < r \cdot u_N$.
Fix an ultralarge m and for any k,

$$u_{m+k} < r^k \cdot u_m$$

u_m is not ultralarge: since the sequence is decreasing at least for ultralarge values of the index and the first term is observable, either it is always decreasing (and positive) so it is not ultralarge or it first increases, but then it will have a maximum which is observable, so it is not ultralarge.

Then

$$\sum_{k=m}^{N} u_k < \sum_{k=m} r^k \cdot u_m = u_m \cdot \sum_{k=m}^{N} r^k$$

This last sum is a geometric series, hence it converges.

The sum $\sum_{k=0}^{\infty}$ is cut in two parts: a finite sum and an infinite one, $\sum_{k=0}^{m-1} + \sum_{k=m}^{\infty}$
For the first part, the context is extended to m.
For all observable n, by closure, u_n is observable. So the sum of an observable number of observable values is observable.
The second part converges so the whole sum converges in the extended context containing m. But the convergence does not depend on the choice of m. So simply: it converges.

Definition 13
*A series $\sum_{k\geq n} u_n$ (or $(u_n)_{n\geq k}$) is **an alternating series**, if $u_n \cdot u_{n+1} < 0$ for each $n \geq k$.*

Theorem 9
Let $(u_n)_{n\geq k}$ be an alternating series decreasing in absolute value. If $\lim_{n\to\infty} u_n = 0$ then $\sum_{n\geq k} u_n$ converges.

Example: This shows that the harmonic alternating series defined by

$$\sum_{n\geq 1} (-1)^n \frac{1}{n}$$

converges.

Exercise 28

Show that if one considers the series $\sum_{k=1}^{\infty}(-1)^k$ then by rearranging the order of the terms, the sum can be made to be equal to any positive or negative number.

! This is a crucial point. A never ending series can yield strange things! Because it never ends. This is why it is important to work on the partial sums. (More difficult theorems state under what conditions can the terms of a series be rearranged without changing the result.)

5 Taylor Series

The idea of this part is to represent a function by a series $\sum_{n \geq k} a_n \cdot (x-c)^n$ such that the series converges to $f(x)$ for some values of x around a point c. This is called the **Taylor series for f at c.**

We first define the nth derivative of f by induction on n.

Definition 14

*Let f be a function. We say that f is **differentiable once at** x if $f'(x)$ exists. We write $f^{(1)}(x) = f'(x)$. By induction, for a positive integer n, we say that f is **differentiable** $n+1$ **times at** x if the function $f^{(n)}$ is differentiable at x. We write $f^{(n+1)}(x) = (f^{(n)})'(x)$.*

Theorem 10

Let N be a positive integer and let $c \in \mathbb{R}$. Let f be a function differentiable $N+1$ times on an open interval containing c and let x be in this interval. Then

$$f(x) = \sum_{n=0}^{N} \frac{(x-c)^n}{n!} \cdot f^{(n)}(c) + \int_c^x \frac{(x-t)^N}{N!} \cdot f^{(N+1)}(t) \cdot dt.$$

Exercise 29

Using any of the convergence criteria, prove that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for any value of x.

Since it converges and depends on x, we define

$$f : x \mapsto \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Exercise 30

Prove the following theorem.

Theorem 11

The number e satisfies

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right).$$

Exercise 31

Same idea for the product; show that this "infinite product" converges, then differentiate it.

$$g : x \mapsto \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Exercise 32

Show that

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2 + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}$$

Definition 15

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Exercise 33

Compute the first partial sums for e^x with $x = 1$ and other values of x and compare with the e^x value of your calculator.

Exercise 34

From complex numbers, recall that $e^{ix} = \cos x + i\sin x$

(1) Write the beginning of the series for e^{ix}

(2) Write the series for $\cos(x)$ and $\sin(x)$ (Think about the real part and imaginary part separately).

(3) Prove that these series converge.

(4) Calculate $\cos(1)$ using this series.

(5) Calculate $\tan(1)$.

We have already shown that the alternating harmonic series converges. Now we can show more.

Exercise 35

Prove that the alternating harmonic series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ converges to $\ln(2)$ i.e.,

$$\ln(2) = \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n}\right).$$

Theorem 12

Let f be a function infinitely many times differentiable on an open interval containing c and let x be in that interval. Suppose that there exists an M such that for each positive integer n the function $f^{(n)}$ is bounded by M on $[x;c]$ (or $[c;x]$ if $c < x$). Then the series

$$\sum_{n\geq 0} \frac{(x-c)^n}{n!} \cdot f^{(n)}(c) \text{ converges to } f(x).$$

Exercise 36

For each of the following, calculate the first terms of the Taylor series. Use induction to obtain the general term. Prove that it converges. Use $c = 0$ for all three.

(1) $\cos(x)$

(2) $\sin(x)$

(3) $\arctan(x)$

Example: A Taylor series for f may converge everywhere without converging to the function f. Consider f given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

One can show that

$$f^{(n)}(0) = 0, \quad \text{for each positive integer } n.$$

The power series $\sum_{n\geq 0} 0 \cdot x^n$ converges to the function which is everywhere 0 and not to f. This is not a contradiction to Theorem 12, as for each $x \neq 0$ and each M there exist n and ξ between 0 and x such that $|f^{(n)}(\xi)| > M$, so the assumptions of the theorem are not satisfied.

Exercise 37

Calculate the Taylor series for $\sqrt{x}$. You must first find a good value for c, which might mean trying several values.

Does it converge for all values of x? (Try using to compute square roots of 0,1,4...)

If it does not converge for, say, 10, is it possible to use another value for c?

Practice exercise 1 Answer page 21

For the following, find the partial sums, determine whether the series converges and find the sum when it exists.

(1) $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \left(\frac{1}{3}\right)^n + \ldots$

(2) $1 + \frac{3}{4} + \frac{9}{16} + \cdots + \left(\frac{3}{4}\right)^n + \ldots$

(3) $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{24}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) + \ldots$

(4) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \ldots$ Hint: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

(5) $1 - 2 + 4 - 8 + \cdots + (-2)^n + \ldots$

(6) $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \cdots + \frac{2n+1}{n^2(n+1)^2} + \ldots$ Hint: $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

(7) $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} + \ldots$

(8) $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \cdots + \frac{(-1)^{n-1} \cdot n}{2n+1}$

(9) $\frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots + \frac{1}{3n+1} + \ldots$

(10) $\ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n) + \ldots$

Practice exercise 2 Answer page 21

For the following, the general term of the series is given. Test the corresponding series for convergence:

(1) $\frac{3n-7}{10n+9}$

(2) $\frac{5}{6n^2+n-1}$

(3) $\frac{\sqrt{n}}{1+2\sqrt{n}+n}$

(4) $n \cdot e^{-n}$

(5) $\frac{5^n}{3^n+4^n}$

(6) $\frac{n^n}{(n!)^2}$

(7) $\frac{2^n \cdot n!}{n^n}$

(8) $\frac{1}{\ln(n)}$

(9) $\frac{n^2}{2^n}$

(10) $\frac{\ln(n)}{n}$

Practice exercise 3 Answer page 22

Give the Taylor series for the following. State for which values of x they converge.

(1) $\dfrac{1}{1-x}$

(2) $\dfrac{1}{1+x}$

(3) $\dfrac{1}{1-2x}$

(4) $\ln(1-x)$

(5) $\dfrac{1}{1+x^2}$

(6) e^{-x}

(7) e^{-x^2}

(8) $\displaystyle\int_0^x e^{-t^2}\,dt$

(9) $\ln\left(\dfrac{1+x}{1-x}\right)$

(10) $(1+x)^p$ for fixed p.

Answers to practice exercice 1, page 18

(1) $\frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$. Converges to $\frac{3}{2}$.

(2) $4\left(1-\left(\frac{3}{4}\right)^n\right)$. Converges to 4.

(3) Rewrite as telescoping series: $1-\frac{1}{(n+1)!}$. Converges to 1.

(4) $1-\frac{1}{n+1}$. Converges to 1.

(5) If n is even: $-n/2$. If n is odd: $n/2+1/2$. Diverges.

(6) Rewrite as telescoping series: $1-\frac{1}{(n+1)^2}$. Converges to 1.

(7) $\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$. Converges to $\frac{1}{2}$.

(8) Diverges because $\lim_{n\to\infty} a_n = \frac{1}{2}$.

(9) Diverges. $3n+1 < 4n$ (for $n>2$) hence $\frac{1}{3n+1} > \frac{1}{4n}$ and $\sum_{n=1}^{N}\frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{N}\frac{1}{n}$ which diverges hence the series is bounded below by a diverging series and diverges also.

(10) Diverges because for ultralarge N, $\ln(N) \not\simeq 0$. Or simply: the terms are increasing and positive.

Answers to practice exercice 2, page 19

(1) Diverges since for ultralarge N we have $\frac{3N-7}{10N+9} = \frac{3-7/N}{10+9/N} \simeq \frac{3}{10} \not\simeq 0$

(2) Converges. By comparison: $\frac{5}{6n^2+n-1} < \frac{5}{6}\frac{1}{n^2}$ (for $n>2$)

(3) Diverges. $\frac{n^{1/2}}{(n^{1/2}+1)^2} = \left(\frac{n^{1/4}}{n^{1/2}+1}\right)^2 = \left(\frac{1}{n^{1/4}+n^{-1/4}}\right)^2 > \left(\frac{1}{4n^{1/4}}\right)^2 = \frac{1}{2n^{1/2}} > \frac{1}{2n}$

(4) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \left(1+\frac{1}{n}\right)\frac{1}{e} < 1$

(5) Diverges. By ratio test: $\frac{5(3^2+4^n)}{3\cdot 3^n+4\cdot 4^n} > \frac{5(3^2+4^n)}{3\cdot 3^n+3\cdot 4^n} = \frac{5}{3} > 1$.

(6) Converges. By ratio test: $\frac{(n+1)^n}{(n+1)n^n} = \frac{(n^n(1+1/n)^n}{(n+1)n^n} \simeq \frac{e}{n+1} \simeq 0$.

(7) Converges. By ratio test: $\frac{(n+1)^2(n+1)n!n!}{n!(n+1)(n+1)!n^n} = \frac{(n+1)^{n-1}}{n^n} = \frac{1}{n}\left(1+\frac{1}{n}\right)^{n-1} \simeq \frac{1}{n}e \simeq 0$

(8) Diverges. By comparison: $\ln(n) < n$ hence $\dfrac{1}{\ln(n)} > \dfrac{1}{n}$ and the harmonic series diverges.

(9) Converges. Ratio test: $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{1}{2} < 1$.

(10) Diverges. Integral test: by setting $\ln(x) = u$ we get $\displaystyle\int_1^\infty \frac{\ln(x)}{x}dx = \frac{1}{2}\ln^2(x)\Big|_1^\infty$ which diverges. By horizontal shifting, this function is below the series.

Answers to practice exercice 3, page 19

(1) $1 + x + x^2 + x^3 + x^4 + \dots$ for $|x| < 1$

(2) $1 - x + x^2 - x^3 + x^4 - \dots$ for $|x| < 1$

(3) $1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \dots$ for $|x| < 1/2$

(4) $-x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \dots$ for $|x| < 1$

(5) $1 - x^2 + x^4 - x^6 + x^8 - \dots$ for $|x| < 1$

(6) $1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \dots$ for all x

(7) $1 - x^2 + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \dfrac{x^8}{4!} \dots$ for all x

(8) $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5 \cdot 2!} - \dfrac{x^7}{7 \cdot 3!} + \dfrac{x^9}{9 \cdot 4!} - \dots$ for all x

(9) $2x + \dfrac{2x^3}{3} + \dfrac{2x^5}{5} + \dfrac{2x^7}{7} + \dfrac{2x^9}{9} + \dots$ for $|x| < 1$

(10) $1 + px + \dfrac{p(p-1)}{2!}x^2 + \dfrac{p(p-1)(p-2)}{3!}x^3 + \mathtt{C}_4^p x^4 + \dots$ for $|x| < 1$

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