Differential Equations

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1 Initial problems

Exercise 1

Radioactivity is due to the decay of nuclei in the atoms. The following observation has been made: For a given element, it takes the same time for half of the nuclei to disintegrate, independently of how much matter there is. Which means that there is a higher activity if there is more matter. This leads to the conclusion that the activity depends on the quantity of matter. It is stated in the law:

"The rate of decay (with respect to time) of the nuclei of radioactive substances is proportional to the number of remaining nuclei."

Let y be the number of remaining nuclei and t is for time.

Write the equation which corresponds to the statement of the law.

Find y as function of t.

Exercise 2

The velocity of a falling body is proportional to the time t during which it has already fallen.

Write the equation corresponding to this statement and show how to find the position of this falling body, with respect to time.

What is the equation for the position of the falling body, with respect to time?

Exercise 3

We have all observed that if something is very hot (a cake coming out of the oven...) its temperature decreases quite quickly in terms of degrees per minute, whereas an object hardly warmer than the surrounding temperature will not lose as many degrees during the first minute.

Newton's (experimental) law of cooling states (in modern units) that the rate at which a body at $T°C$ above surrounding temperature cools is proportional to T

Find the mathematical equivalent to this law.

A body at $68°C$ in a room at $16°C$ is $55°c$ after 5 minutes. What will its temperature be in another 5 minutes?

What will its temperature be in yet another 5 minutes?

After how long will the body reach the surrounding temperature? Comment your answer.

Definition 1

A differential equation is an equation involving an independent variable t and the derivatives of y with respect to t, where the function *y is the unknown.*

Definition 2 (First order differential equation)

A first order differential equation involves only the first derivative of y:

$$y' = f(y, t)$$

! The solution to a differential equation is a function or a whole class of functions.

The independent variable is traditionally t rather than x. This is because differential equations appeared first in physics problems related to time.

2 Slope Fields

If a graph is drawn with t as horizontal coordinate and y as vertical coordinate, at $\langle t, y \rangle$ the differential equation yields a specific slope $f(y, t)$. A visualisation can be provided by marking some of the slopes on a grid.

Exercise 4

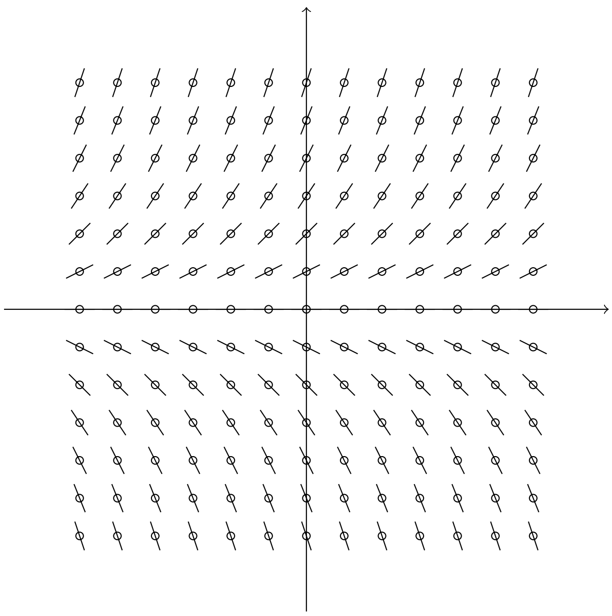
In the following drawings, each arrow represents a slope. They form a slope-field. (One must imagine that between two arrows the slope changes continuously).

- Choose a point on the plane, follow the arrows with a pencil to draw a curve. What does this curve represent?

  The curves that can be drawn as above are called **specific solutions**

- Repeat this with some other points.

(1)



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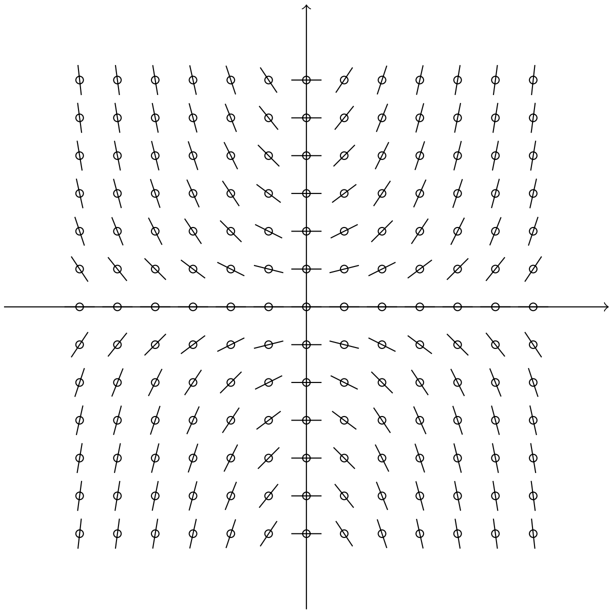

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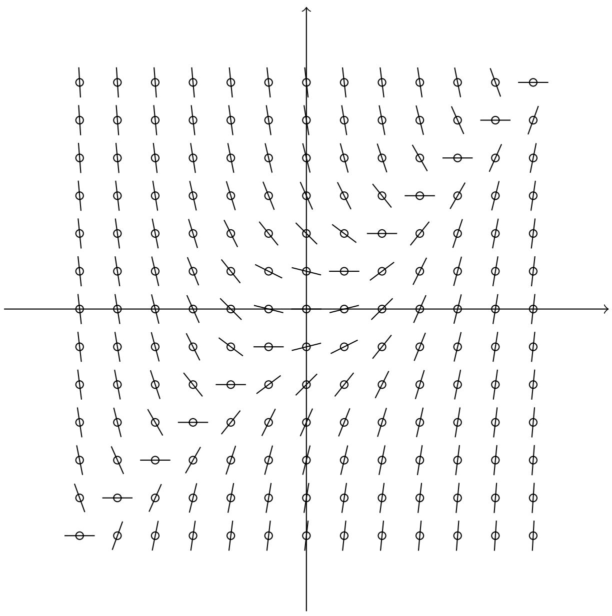
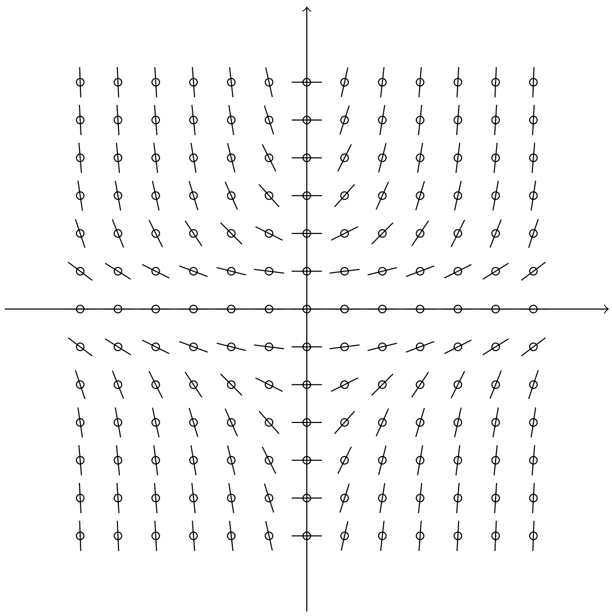


Exercise 5

The following slope-field is for $y' = y^2 \cdot t$



Follow the solutions for different initial values.

Exercise 6

Draw the slope-field for

$$y' = 2t\sqrt{1-y^2}$$

for $t \in [-1; 1] \quad y \in [-1; 1]$ (grid every 0.5)

Exercise 7

Draw the slope-field for

$$y' = (1+y^2) \cdot e^t$$

for $t \in [-1; 1] \quad y \in [-1; 1]$ (grid every 0.5)

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3 Specific solutions

The following method enables to sketch rapidly the general form of the family of curves which are solution of any equation of the form $y' = f(t, y)$, under the condition that y'' exists.

Example:

Consider the equation

$$y' = t(y-1)$$

Differentiating again leads to

$$y'' = (y-1) + t \cdot y' = (y-1) + t \cdot t(y-1)$$

$$y'' = (t^2+1)(y-1)$$

We know that the first derivative gives information about the slope and the second derivative gives information about the bending.

The first factor t^2+1 is positive. The second is positive if $y > 1$. This means that if $y > 1$ the second derivative is positive and the curve bends upwards, if $y < 1$ the curve bends downwards and if $y = 0$ there is no bending, hence the curve is a straight line. As for $y = 0$ we have that $y' = 0$ the line is horizontal.

The original equation also yields $y' = 0$ for $t = 0$

It is also possible to see that the closer to $y = 1$ the closer to zero y' will be, hence the curve will be flatter.

Exercise 8

Draw the $y' = 0$ lines. Draw the $y'' = 0$ lines. Sketch some of the specific solutions.

Exercise 9

$$y' = y + e^t$$

leads to $y'' = y' + e^t = y + 2e^t$

The locus of inflexions ($y'' = 0$) is the curve $y = -2e^t$.

- What are the values of y' when t takes large positive values? When t takes large negative values?
- Draw the fixed points curve and inflexion curve.
- Draw some of the specific solutions.

Exercise 10

Sketch some of the specific solutions of the following:

(1) $y' = y(1-t)$

(2) $y' = t^2 y$

(3) $y' = y + t^2$

(4) $y' = t + e^y$

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4 Solving

Reminder: The differential dy is the first order approximation (along the tangent line). If y is a function of t, then

$$dy = y' \cdot dt$$

and

$$y = \frac{dy}{dt}$$

Definition 3 (First Order)
If only the first derivative appears in the equation, then it is a First Order Differential Equation

$$y' = \frac{dy}{dt} = f(t, y)$$

Definition 4 (Separable Variables)
If the differential equation is of the form

$$\frac{dy}{dt} = f(t) \cdot g(y)$$

then it is a differential equation with separable variables

When solving differential equations we consider $df(x)$ and dx as "separable". The easy way to understand this is to imagine that dx is an infinitely small quantity and that $df(x)$ is the corresponding variation of the function along the tangent line to f at point $< x; f(x) >$

Definition 5 (Differential)
The derivative of $f(x)$ with respect to x is noted either $f'(x)$ or

$$\frac{df(x)}{dx}$$

.
The differential is

$$df(x) = f'(x) \cdot dx$$

Definition 6
An initial value is a given value for t and the corresponding value for y generally in the form

$$y(t_0) = y_0$$

Exercise 11
Find the general solutions of the following:

(1) $y' = y \cdot (y + 1)$

(2) $y' = t \cdot \sin(t^2)$

(3) $y' = e^{-y}$

(4) $y' = y \cdot \tan(t)$

(5) $(t - 1) \cdot y' - 2y = 0$

(6) $y' = t \cdot y \cdot \ln(t)$

Exercise 12
Solve the initial value problems:

(1) $y' = y^2 \cdot t^2$ initial value: $y(1) = 2$

(2) $y' = t \cdot \sqrt{y}$ initial value $y(0) = 3$

(3) $y' = \dfrac{\ln(t)}{y}$ initial value $y(1) = -2$

(4) $y' = t \cdot y - y + 2t - 2$ initial value $y(0) = 0$

(5) $y' = (y^2 - 3y + 2) \cdot \sqrt{t}$ initial value $y(1) = 2$

Exercise 13
Let Γ_k be a given differentiable curve such that the tangent through each point $< x; y >$ of Γ_k intercepts the vertical axis at $< 0; k \cdot y >$.

Which are the curves that satisfy this property? (What can be said of k=0, k=1)

(You will need to write the general formula for the equation of the straight line tangent to Γ_k at $< x_0; y_0 >$ using y as dependent variable of the curve Γ_k)

Definition 7 (First Order Homogeneous Linear DE)
An equation of the form

$$y' + p(t) \cdot y = 0$$

is called a First Order Homogeneous Linear Differential Equation.
Linear, because y' and y only appear linearly.
Homogeneous, because the right hand side of the equation is zero.

Exercise 14

Prove the following theorem:

Theorem 1
Let $P(t)$ be an antiderivative of $p(t)$, then

$$y' + p(t) \cdot y = 0$$

has the following solution:

$$y(t) = C \cdot e^{-P(t)}$$

Exercise 15

A bacterial culture grows at a rate proportional to its population. If it has a population of one million at time $t = 0$ and 1.5 million at $t = 1h$, find its population as a function of t

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Exercise 16

A radioactive element decays with a half life of 6 years. Starting with 10kg at time $t = 0$, find the amount of initial matter as a function of t

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Exercise 17

Solve the initial condition problem:

$$y' + y \cdot \cos(t) = 0$$

(1) with $y(0) = \frac{1}{2}$

(2) with $y(\pi/4) = \frac{1}{2}$

Make a rough sketch of the two curves.

Exercise 18

Solve

$$y' + 3t^{-1}y = 0$$

with $y(1) = 2$

Check what is happening in the neighbourhood of $t = 0$

Make a rough sketch of the solution.

Exercise 19

Note that a homogeneous DE always has $y = 0$ (the constant function 0) as a solution. Thus in the following, non trivial solutions are required.

Solve:

(1) $y' + 5y = 0$

(2) $y' + \frac{y}{1+t^2} = 0$

(3) $y' - 2y = 0$

(4) $y' + t^2 y = 0$

Exercise 20

Solve the initial condition equations:

(1) $y' + y = 0$ initial value: $y(0) = 4$

(2) $y' + y\sin(t) = 0$ initial value: $y(\pi) = 1$

(3) $y' + y\sqrt{1+t^2} = 0$ initial value: $y(0) = 0$

(4) $y' + y\cos(e^t) = 0$ initial value: $y(0) = 0$

(5) $ty' - 2y = 0$ initial value: $y(1) = 4 \quad t > 0$

(6) $t^2 y' + y = 0$ initial value: $y(1) = -2 \quad t > 0$

(7) $t^3 y' = 2y$ initial value: $y(1) = 1 \quad t > 0$

(8) $t^3 y' = 2y$ initial value: $y(1) = 0 \quad t > 0$

(9) $y' - 3y = 0$ initial value: $y(1) = -2$

(10) $y' + ye^t = 0$ initial value: $y(0) = e$

Definition 8 (First Order Linear Differential Equation)
An equation of the form

$$y' + p(t) \cdot y = f(t)$$

is called a First Order Linear Differential Equation.

Example: A population has a net birthrate $b(t)$. Hence its rate of variation will depend on the size of the population and the birthrate: $b(t) \cdot y$.

There is also a net immigration rate $f(t)$ which only depends on time.

Therefore the variation of population will be

$$y' = b(t) \cdot y + f(t)$$

The idea is to make the left hand side look like the derivative of a product. Multiply both sides by a function w (function of t: in fact $w(t)$) - called an **integrating factor:**

$$w \cdot y' + w \cdot p \cdot y = w \cdot f$$

Exercise 21

What should w be so that $w(t) \cdot y'(t) + w(t) \cdot p(t) \cdot y(t) = (w(t) \cdot y(t))'$?
(Do not forget the chain rule!)

Once w is determined, we have

$$wy = \int w(t)f(t)dt + C$$

dividing by w gives the solution.

Exercise 22

Use this method to solve

$$y' + 3t^{-1} \cdot y = t^2$$

for $t > 0$ and $y(1) = \frac{1}{2}$

Exercise 23

A population has a net birthrate of 2% per year and a net immigration rate of $100000 \sin(t \cdot 2\pi)$ (The sine function reproduces the seasonal variation). At $t = 0$ the population is 10^6.

Find the population as a function of t.

Exercise 24

Find the general solution of the differential equation:

(1) $y' + 4y = 8$

(2) $y' - 2y = 6$

(3) $y' + ty = 5t$

(4) $y' + e^t y = -2e^t$

(5) $y' - y = t^2$

(6) $2y' + y = t$

(7) $ty' + 2y = 1/t \quad t > 0$

(8) $ty' + y = \sqrt{t} \quad t > 0$

Exercise 25

Writing equations:

(1) Consider the following problem proposed by Claude Perrault, from Paris in 1674. It describes the path of a silver pocket watch which is pulled by its chain. The curve is called the “tractrix.”

Take a chain-watch and pull it by the chain, the end of the chain following the straight edge of the table. If the starting position of the watch is when the chain is perpendicualr to the table, at the exact distance if the length of the chain, what will be the equation of the path of the watch?

a is the length of the watch chain (constant). d can be calculated with respect to y and x and a

The chain is tangent to the curve of the path, because the watch is pulled in the direction of the chain.

(a) Write the (differential) equation for y. In fact, it will be x with respect to y.

(b) This is not easy to solve, but check that

$$x = -\sqrt{a^2 - y^2} - a \cdot \ln\left(\frac{a - \sqrt{a^2 - y^2}}{y}\right)$$

is the solution.

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5 Numeric approximation: the Euler method

Very often, an explicit solution cannot be found.

Let $y' = f(t; y)$ with $y(a) = y_0$

then $Y(a + \Delta t) = y_0 + f(a; y_0) \cdot \Delta t$ where $Y(x)$ is an approximation of the exact solution $y(s)$

Exercise 26

(1) Write $Y(a + 2\Delta t)$

(2) Write $Y(a + k\Delta t)$ using the $\sum$ sign.

(3) Show that if $a + k\Delta t = s$ then

$$Y(s) = y_0 + \sum_{t=a}^{s} f(t; y(t)) \cdot \Delta t$$

Exercise 27

The equation $y' = t - y^2$ is not linear, not separable.

Compute the Euler method for $y(0) = 0$, $\Delta t = 0.2$ and $t \in [0; 1]$

Sketch the curve.

Exercise 28

Same equation and initial value. $\Delta t = 0.1$ Compute and sketch the curve for the same interval.

Exercise 29

Compute the Euler approximation for $y' = \dfrac{t \sin(t)}{y} \quad y(2) = 5$ for $t \in [2; 8]$

(Use of a spreadsheet may help.)
