# Analysis
with ultrasmall numbers

Higher Level

Collège André-Chavanne
Genève

*Infinity itself looks flat and uninteresting. [...] The chamber [...] was anything but infinite, it was just very very very big, so big that it gave the impression of infinity far better than infinity itself.*
*(Douglas Adams: The Hitchhiker's Guide to the Galaxy)*

# Contents

# 1
# Velocity and Position

**Exercise 1**

Suppose the velocity [1] of a car is constant and equal to $60km/h$.

(1) Let $f$ be the function which describes the position of the car with respect to time.
Draw the graph $f$ for $t$ ranging from 0 to 3 hours.

(2) Let $v$ be the function which describes the velocity of the car with respect to time.
Draw the graph of $v$ for $t$ ranging from 0 to 3 hours.

(3) Given the position graph, how can one find the velocity of the car at any given time?

(4) Given the velocity graph, how can one find the position of the car after any given time?

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⚠ Note the difference: velocity (deduced from position) is *local*. It is possible to give the velocity *at* a given time. Position (deduced from velocity) is *global*. It is only possible to find the *variation* of the position over an *interval* of time.

**Exercise 2**

The velocity of a car (in km/h) is given by the following function with respect to time (in h): (decimal division of hours for simplicity)

$$v : t \mapsto \begin{cases} 60 & \text{if } 0 \leq t \leq 0.5 \\ 120 & \text{if } 0.5 < t \leq 2 \\ 80 & \text{if } 2 < t \leq 2.5 \\ 60 & \text{if } 2.5 < t \leq 3 \end{cases}$$

Calculate the positions at $t = 1, \quad t = 2$ and $t = 3$.

Draw the velocity graph and indicate *on the velocity graph* where the position at $t = 2$ can be drawn.

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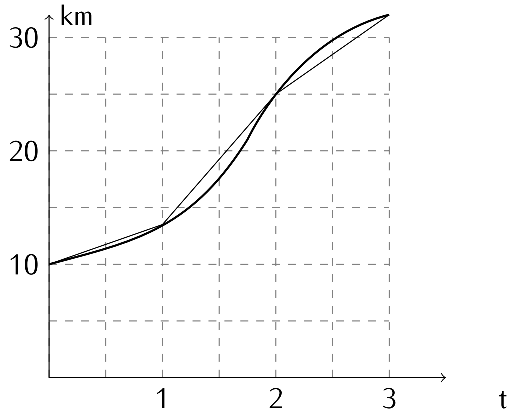
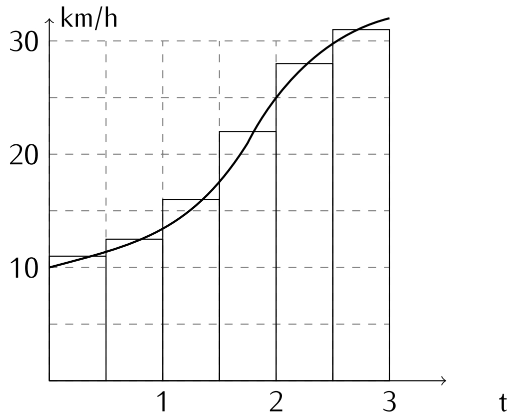
[1] The velocity is speed with a direction. Speed is always positive (or zero); velocity can be negative.

The following curve can be approximated by a piecewise linear function whose slope is easily calculated by pieces. If this curve represents the position function of a moving body, the linear pieces may given approximate representation of the velocity function.




The following area under a curve can be approximated by a "staircase" function whose area is calculated by adding the areas of the rectangles. If this curve represents the velocity function of a moving body, the rectangles may give an approximate representation of the position function.




The main goal of the subject called **mathematical analysis** will be to check when and how to approximate a curve by pieces of straight lines and when and how to approximate areas by rectangles and to understand what these can be used to calculate. Intuitively, it should seem clear that in order for the approximation to be good, the pieces of straight lines or the rectangles must be small – or that the number of pieces is large. The crucial questions are: How small? and How large?

# 2

# Basic Principles

**Exercise 3**

Hold a pencil in your hand. Do not move.
Now drop the pencil.

First the pencil was motionless. Then it was in motion.
How did the motion start? How is the transition from "not moving" to "moving"?

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**Exercise 4**

If $\delta$ is a positive value which is extremely small (even smaller than that!),

(1) what can you say about the size of $\delta^2$, $2 \cdot \delta$ and $-\delta$?

(2) what can you say about $2 + \delta$ and $2 - \delta$?

(3) what can you say about $\frac{1}{\delta}$?

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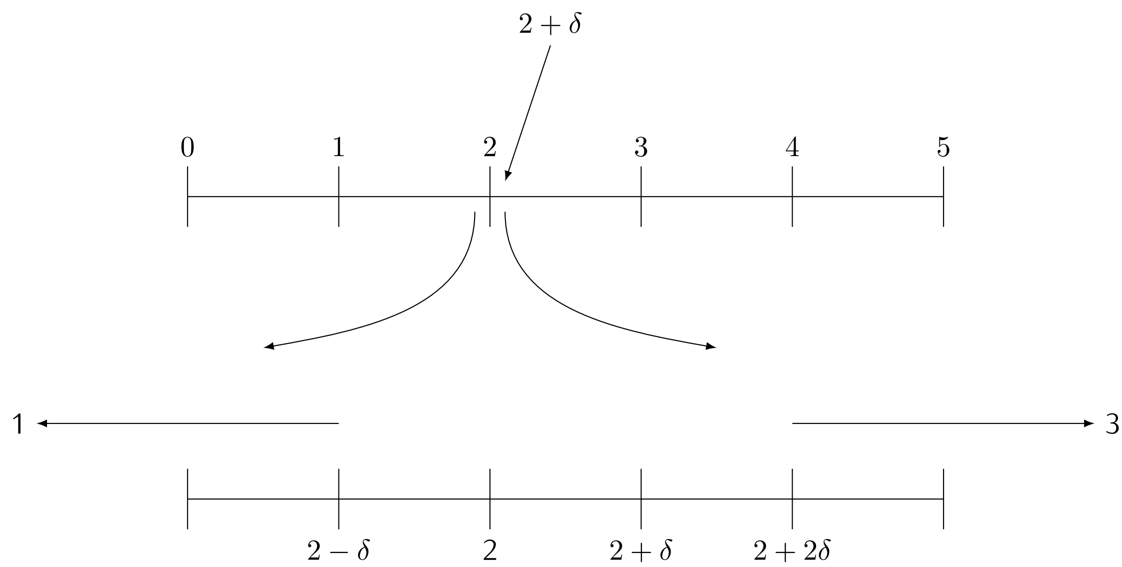


Zooming in

**Exercise 5**

If $N$ is a positive huge number (really very huge!),

(1) what can you say about $N^2$, $2N$ and $-N$?

(2) what can you say about $N+2$ and $N-2$?

(3) what can you say about $\frac{1}{N}$?

(4) what can you say about $\frac{N}{2}$?

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**Exercise 6**

Let $f : x \mapsto x^2$, and let $\delta$ be "vanishingly small" and positive.

(1) Draw the result of a zoom on $f$ centred on $\langle 2; 4 \rangle$ so that $\delta$ becomes visible.
Show, on the drawing, the values 2 and $f(2)$, $2+\delta$ and $f(2+\delta)$, $2-\delta$ and $f(2-\delta)$.
What does the curve look like?

(2) For the same function, draw the result of a zoom centred on $\langle 1; 1 \rangle$
Show, on the drawing, the values 1 and $f(1)$, $1+\delta$ and $f(1+\delta)$, $1-\delta$ and $f(1-\delta)$.

(3) Similar question for a zoom centred on $\langle 0; 0 \rangle$.

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**Exercise 7**

Draw the result of zooms so that $\delta$ becomes visible for
$g : x \mapsto x^3$, and $h : x \mapsto |x|$
For $g$: centres are $\langle 1; 1 \rangle$, $\langle 2; 8 \rangle$ and $\langle 0; 0 \rangle$
For $h$: centres are $\langle 1; 1 \rangle$, $\langle 2; 2 \rangle$ and $\langle 0; 0 \rangle$

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**Exercise 8**

Draw a zoom centred on $\langle 0; 0 \rangle$ and another zoom centred on $\langle 0; -1 \rangle$ for

$$k : x \mapsto \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

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When we say that $\delta$ is "tiny", we want it to be tiny compared to all the parameters involved; this leads to the following definition:

**Definition 1**
*The **context** of a property, function or set is the list of parameters used in its definition. The context can be a single number.*

A context is *extended* if parameters are added to the list.

Before defining more precisely what it means to be "tiny" we must first clarify what it means to be observable:

**Observability**

(1) Numbers defined without reference to observability are always observable – or standard.

(2) If $a$ is not observable in the context of $b$, then $b$ is be observable in the context of $a$. (the context from which both are observable is the common context).

(3) **Closure:** If a number satisfies a given property, then there is an observable number satisfying that property.

(4) A property referring to observability is true if and only if it is true when its context is extended.

A consequence of (3) is that the results of operations between two numbers are in their common context.

The word "observable" , by convention, refers to a context. Informally: the context is the parameters, sets and functions the statement is about. Therefore to determine the context of a statement, one must be able to understand it and describe what it says and about what it says something.

But: a consequence of (4) is that it does not matter what the context is precisely provided it contains at least all parameters involved.

All "familiar" numbers such as $1;\ 3;\ 10^{10};\ \sqrt{2}$ or $\pi$ are always observable, or standard, but also – in general –

**Theorem 1**
*$f(a)$ is observable.*

This refers to the context, by the word "observable". The only parameters of this property are $f$ and $a$. This is the context.

Non observable values do not show up unless explicitly summoned.

**Definition 2**
*A real number is **ultrasmall** if it is nonzero and smaller in absolute value than any strictly positive observable number*

This definition makes an implicit reference to a context.

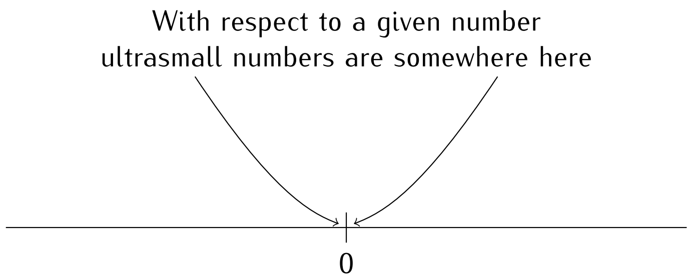
⚠ Note that $0$ is not ultrasmall.

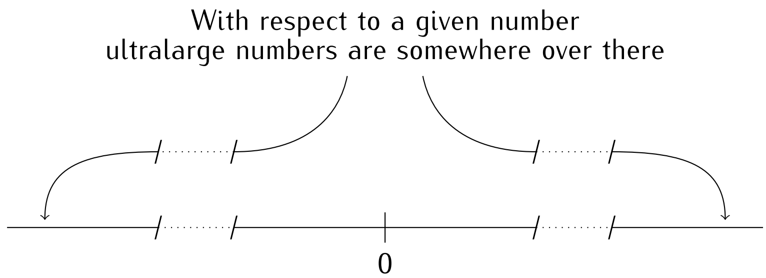
**Principle of ultrasmallness**
*Relative to any context, there exist ultrasmall real numbers.*

Such an ultrasmall number is then part of an **extended context**.
Given a context, if $\varepsilon$ is ultrasmall then $\varepsilon$ is not observable.

**Definition 3**
*A real number is* ***ultralarge*** *if it is larger in absolute value than any strictly positive observable number*

Note the asymmetry: if $h$ is ultrasmall relative to $x$, then it is not observable. But $x$ is observable relative to $h$ (see the third item of the observability pricniple), hence $x$ **is not** ultralarge relative to $h$.




**Definition 4**
*Let $a, b$ be real numbers. We say that $a$ is* ***ultraclose*** *to $b$, written*

$$a \simeq b,$$

*if $b - a$ is ultrasmall or if $a = b$.*

This definition makes an implicit reference to a context.
In particular, $x \simeq 0$ if $x$ is ultrasmall or zero.

If $a \simeq b$ then $a$ and $b$ are said to be neighbours. If $a$ is a neighbour of $b$ and is observable (relative to some context) then $a$ is the observable neighbour of $b$.

**Theorem 2**
*Relative to a context: If $a$ and $b$ are observable and $a \simeq b$, the $a = b$.*

**Exercise 9**

Prove the previous theorem. (you will need to refer to closure)

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A rational number may have an observable neighbour which is not rational. The number $\sqrt{2}$ is always observable because it is completely and uniquely defined by the parameter $2$. Relative to this context consider an ultralarge $N$ and take the first $N$ digits of $\sqrt{2}$. This is a rational number which is not observable. Yet it is ultraclose to an observable number which is $\sqrt{2}$.

The existence of an observable neighbour is given by the following

**Principle of the observable neighbour**
*Relative to a context, any real number $x$ which is not ultralarge can be written in the form $a + h$ where $a$ is observable and $h \simeq 0$.*

**Exercise 10**

Show that if $x$ has an observable part, then it is unique.

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This unique number is **the observable neighbour** of $x$.

**Exercise 11**

Prove the following:

**Theorem 3**
*Let $[a;b]$ be an interval. Show that if $x$ is in $[a;b]$, then the observable part of $x$ is not outside $[a;b]$.*

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**Exercise 12**

Prove the following:

(1) If $\varepsilon$ is ultrasmall relative to $x$ then $\frac{1}{\varepsilon}$ is ultralarge relative to $x$.

(2) If $M$ is ultralarge relative to $x$ then $\frac{1}{M}$ is ultrasmall relative to $x$.

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**Exercise 13**

Prove the following theorems (together they give all the rules needed for analysis and will be referred to by "ultracomputation" or "ultracalculus"):

**Theorem 4**
*Let $\varepsilon$ and $\delta$ be ultrasmall relative to a context and let $a$ be observable and not zero.*

*(1) Then: $a \cdot \varepsilon$ is ultrasmall.*

*(2) Then: $\varepsilon + \delta \simeq 0$*

*(3) Then: $\varepsilon \cdot \delta$ is ultrasmall*

*(4) If $a \neq 0$ Then: $\dfrac{a}{\varepsilon}$ is ultralarge*

**Theorem 5 (Ultracomputation)**
*Relative to a context: If $a$ and $b$ are observable and not zero and if $a \simeq x$ and $b \simeq y$,*

*(1)* $a + b \simeq x + y$

*(2)* $a - b \simeq x - y$

*(3)* $a \cdot b \simeq x \cdot y$

*(4) If also* $b \neq 0$, $\dfrac{a}{b} \simeq \dfrac{x}{y}$.

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For the last item of theorem 5, it is enough to show:

Relative to a context. If $b$ is observable and $b \neq 0$ and if $b \simeq y$ then $\dfrac{1}{b} \simeq \dfrac{1}{y}$

and use item 3 to conclude.

**Practice exercise 1** Answer page 15
Consider a context.

(1) Give an example of $x$ and $y$ such that $x \simeq y$ but $x^2 \not\simeq y^2$.

(2) Give an example of $x$ and $y$ such that $x \simeq y$ but $\frac{1}{x} \not\simeq \frac{1}{y}$.

**Practice exercise 2** Answer page 15
Relative to a context.
In the following, assume that $\varepsilon, \delta$ are positive ultrasmall and $H, K$ positive ultralarge numbers. Determine whether the given expression yields an ultrasmall number, an ultralarge number or a number in between.

(1) $1 + \dfrac{1}{\varepsilon}$

(2) $\dfrac{\sqrt{\delta}}{\delta}$

(3) $\sqrt{H+1} - \sqrt{H-1}$

(4) $\dfrac{H+K}{H \cdot K}$

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7}$

(6) $\dfrac{\sqrt{1+\varepsilon} - 2}{\sqrt{1+\delta}}$

**Practice exercise 3** Answer page 15
Relative to a context find ultrasmall $\varepsilon$ and $\delta$ (or the relation between them) such that $\dfrac{\varepsilon}{\delta}$ is:

(1) not ultralarge and not ultrasmall,

(2) ultralarge,

(3) ultrasmall.

⚠ The previous exercise show that if no relation is known between ultrasmall numbers $\varepsilon$ and $\delta$, their quotient can be of any possible magnitude.

**Contextual Notation**
The only acceptable properties are those that do not refer to observability or those that use the symbol "$\simeq$".

## Answers to practice exercises

**Answers to practice exercice 1**, page 12

(1) Let $x = N$ be ultralarge, and $y = N + \frac{1}{N}$ so $x \simeq y$ but $x^2 = N^2 \not\simeq N^2 + 2 + \frac{1}{N^2} = y^2$.

(2) Let $h$ be ultrasmall, then let $x = h$ and $y = h^2$. Then $x \simeq 0$ and $y \simeq 0$ hence $x \simeq y$. Then $\frac{1}{h}$ and $\frac{1}{h^2}$ are both ultralarge and $\frac{1}{h^2} - \frac{1}{h} = \frac{1}{h}(\frac{1}{h} - 1)$. By ultracomputation, this is ultralarge, hence $\frac{1}{x} \not\simeq \frac{1}{y}$.

**Answers to practice exercice 2**, page 12

The terms ultrasmall or ultralarge all refer to a given context.

(1) As $\frac{1}{\varepsilon}$ is ultralarge $1 + \frac{1}{\varepsilon}$ is ultralarge.

(2) We have $\frac{\sqrt{\delta}}{\delta} = \frac{1}{\sqrt{\delta}}$ which is ultralarge.

(If $\delta < c$ for any observable $c$, then $\sqrt{\delta} < \sqrt{c}$ and $\sqrt{\delta} \simeq 0$ hence $\frac{1}{\sqrt{\delta}}$ is ultralarge.)

(3) Maybe surprisingly, this is ultrasmall. To see this we multiply and divide by the conjugate:

$$\begin{aligned}
\sqrt{H+1} - \sqrt{H-1} &= \frac{(\sqrt{H+1} - \sqrt{H-1})(\sqrt{H+1} + \sqrt{H-1})}{\sqrt{H+1} + \sqrt{H-1}} \\
&= \frac{(H+1) - (H-1)}{\sqrt{H+1} + \sqrt{H-1}} \\
&= \frac{2}{\sqrt{H+1} + \sqrt{H-1}}.
\end{aligned}$$

$H$ is assumed positive, its square root (plus or minus 1) is also a positive ultralarge. The sum of 2 positive ultralarge numbers is ultralarge hence the quotient is ultrasmall.

(4) $\dfrac{H+K}{HK} = \dfrac{1}{K} + \dfrac{1}{H}$ is ultrasmall.

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7} = \dfrac{35 + 7\varepsilon - 35 - 5\delta}{49 + 7\delta} = \dfrac{\overbrace{7\varepsilon - 5\delta}^{\simeq 0}}{\underbrace{49 + 7\delta}_{\simeq 49}}$ is ultrasmall or zero.

(6) $\dfrac{\overbrace{\sqrt{1+\varepsilon} - 2}^{\simeq -1}}{\underbrace{\sqrt{1+\delta}}_{\simeq 1}} \simeq -1$, hence not ultralarge and not ultrasmall.

**Answers to practice exercice 3**, page 12

(1) Take $\varepsilon = \delta$ then $\dfrac{\varepsilon}{\delta} = 1$.

(2) Take $\delta = \varepsilon^2$, then $\dfrac{\varepsilon}{\delta} = \dfrac{1}{\varepsilon}$ is ultralarge.

(3) Take $\varepsilon = \delta^2$, then $\dfrac{\varepsilon}{\delta} = \delta$ is ultrasmall.

# 3 Derivatives

**Exercise 14**

Let

$$f : x \mapsto x^2$$

The graph of this function is a curve (a parabola). Zoom in on the point $\langle 2, 4 \rangle$. 2 and 4 are always observable. Consider the value of the function at $2+\Delta x$, and draw a straight line passing through $\langle 2, 4 \rangle$ and $\langle 2 + \Delta x, f(2 + \Delta x) \rangle$.

- What is the slope of this straight line?
- What is the observable neighbour of this slope?

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**Definition 5**

*A real function $f$ defined on an interval containing $a$ is **differentiable at** $a$ if there is an observable value $D$ such that, for any $\Delta x$*

$$\frac{f(a + \Delta x) - f(a)}{\Delta x} \simeq D$$

*Then $D = f'(a)$ is the **derivative** of $f$ at $a$.*

The "for any $\Delta x$" means that the value of $D$ must not depend on the choice of the ultrasmall $\Delta x$, in particular, whether it is positive or negative.

When the derivative exists, it is the observable neighbour of $\dfrac{f(a + \Delta x) - f(a)}{\Delta x}$.

⚠ This is a statement about $f$ at $a$, hence the context is the list of parameters of $f$ and $a$.

Metaphorically, finding the derivative can be described by: Zoom in. If what you see is indiscernible from a straight line, then measure the slope of that line. Zoom out. Drop what you cannot see anymore.

**Exercise 15**

Using definition 5 calculate the derivatives (if they exist) of the following:

(1) $f : x \mapsto 3x^2 + x - 5$ at $x = -2$ and $x = 2$.

(2) $g : x \mapsto 2x^3 - 2$ at $x = 1$ and $x = 0$.

(3) $h : x \mapsto |x|$ at $x = 2$, $x = -2$ and at $x = 0$.

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**Exercise 16**

Let $f : x \mapsto x^3 - x - 6$. Check that 2 is a root of $f$. Are there other roots?

At what values of $x$ is the derivative equal to zero? What is the value of the function at these points? At what values of $x$ de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

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**Exercise 17**

Let $f : x \mapsto 2x^3 - 4x^2 + 2x$. At what values of $x$ is the function equal to zero? At what values of $x$ is the derivative equal to zero? What is the value of the function at these points? At what values of $x$ de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

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**Exercise 18**

Consider the derivative at $x$ (general case) of the function

$$f : x \mapsto x^2 + 3x.$$

Show that it is differentiable for all $x$ and that $f'(x) = 2x + 3$.

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Notice that in a derivative, the division is **always** between two ultrasmall numbers. They <u>cannot</u> be replaced by 0 since $\frac{0}{0}$ is not defined.

If a function is differentiable for all $x$ in an interval, then $f$ is said to be differentiable on the interval.

**Definition 6**

*If $f'(x)$ exists for all $x$ in $I$ **the derivative function** is*

$$\begin{aligned} f' : I &\to \mathbb{R} \\ x &\mapsto f'(x) \end{aligned}$$

If $f'(a) = 0$, then in an ultrasmall neighbourhood of $a$ the function is **stationary** – on an ultrasmall neighbourhood $[a - \Delta x; a + \Delta x]$ its variation is of the form $\varepsilon \cdot \Delta x$ for ultrasmall $\varepsilon$ – its graph is indistinguishable from a horizontal line.

**Exercise 19**

Differentiate $f : x \mapsto x^2$ and $g : x \mapsto x^3$ at general $x$.

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**Notation:** Let $\Delta x$ be ultrasmall relative to $f$ and $x$. We write

$$\Delta f(a) = f(a + \Delta x) - f(a) \text{ or } f(a + \Delta x) = f(a) + \Delta f(a).$$

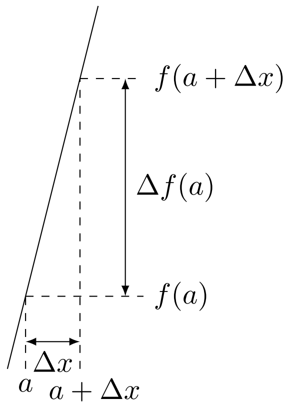
Hence, we have:

$$\frac{\Delta f(a)}{\Delta x} \simeq f'(a).$$

**Notation:** A "$\simeq$" symbol may be replaced by a "$=$" symbol by adding a value ultraclose to zero on one of the sides i.e., $A \simeq B \Rightarrow A = B + \varepsilon$ where $\varepsilon \simeq 0$. Sometimes working with equality is safer.

Hence

$$\frac{\Delta f(a)}{\Delta x} = f'(a) + \varepsilon \text{ with } \varepsilon \simeq 0$$




Note: drawings involving ultrasmall or ultralarge values are not meant to be to scale nor be a correct representation. Their purpose – as all drawings used in mathematics – is merely to help the mind.

**Practice exercise 4** Answer page 27

Using definition 5, give the derivative functions of the following functions:

(1) $f : x \mapsto 3x + 2$

(2) $g : x \mapsto 2x^2 - x$

(3) $h : x \mapsto 5x^3 + 2x^2 - x$

(4) $k : x \mapsto 5x^3 + 2x^2 + 3x + 2$

In some cases, the slope to the right of a point is not the same as the slope to the left of that point. The derivative is the slope when it is the same on both sides.

**Exercise 20**

Let $f : x \mapsto ax + b$.

Show that the slope of $f$ is $a$.

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**Theorem 6 (Derivative at a maximum or a minimum.)**

*Let $f$ be a real function defined on an open interval $]a; b[$ differentiable at $c \in ]a; b[$.*
*If $f(c)$ is a local maximum (or a local minimum) then $f'(c) = 0$.*

**Exercise 21**
Prove theorem 6. (Hint, consider the variation $\Delta f(c)$.)

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## Variation

We now make the link between local variation and derivative.

**Definition 7**
*Let $f$ be a real function defined on an interval $I$.*

(1) *The function $f$ is **increasing on** $I$ if $f(x) \leq f(y)$, whenever $x < y$ in $I$.*

(2) *The function $f$ is **decreasing on** $I$ if $f(x) \geq f(y)$, whenever $x < y$ in $I$.*

If the inequalities are strict, then we say that the function is strictly increasing or strictly decreasing.

**Exercise 22**
Prove the following theorem:

**Theorem 7 (Variation and Derivative)**
*Let $f$ be a real function differentiable at $a$. Then*

(1) *If $f'(a) \geq 0$ ($> 0$) then $f$ is (resp. strictly) increasing at $a$.*

(2) *If $f'(A) \leq 0$ ($< 0$) then $f$ is (resp. strictly) decreasing at $a$.*

(3) *If $f'(x) = 0$ then $f$ is stationary at $a$.*

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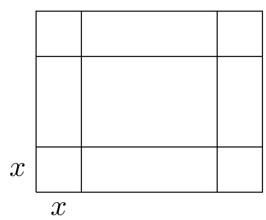
The converse is obvious: if $f$ is increasing at $a$, then $f'(a) > 0$.

**Exercise 23**
A factory wants to make cardboard boxes (with no top) out of sheets of $30cm \times 16cm$




The volume will be a function of $x$. The dimensions of the base are $30 - 2x$ and $16 - 2x$ (in centimetres). The height is $x$. What value(s) of $x$ give(s) the maximum volume for the box?

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**Exercise 24**
Differentiate

(1) $f : x \mapsto \frac{1}{x}$ for $x = 1$ and $x = 2$.

(2) $g : x \mapsto \frac{1}{3x+2}$ for $x = 0$ and $x = 1$.

(3) $h : x \mapsto \frac{1}{x^2}$ for $x = 1$ and $x = -1$.

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## Tangent line

Suppose $f$ is differentiable at $x_0$. We observe that through a microscope, the curve of a function $f$ at $x_0$ is indistinguishable from a straight segment. This straight segment meets the function at $\langle x_0; f(x_0)\rangle$ and there is a (unique) line which extends this segment with slope equal to the derivative. This line is the tangent line.

**Definition 8**
*Let $f$ be differentiable at $x_0$. The tangent line $T_{x_0}$ is a line through $\langle x_0; f(x_0)\rangle$ with slope $f'(x_0)$.*

The tangent line satisfies $T(x_0) = f(x_0)$ and $T'(x_0) = f'(x_0)$.

**Exercise 25**
Let $f : x \mapsto x^2$. Find the equation of the straight line tangent to $f$ at $x = 3$.

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**Exercise 26**
Show that

$$T_{x_0} : x \mapsto f'(x_0)(x - x_0) + f(x_0).$$

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**Exercise 27**
Give the equation of the line tangent to $x \mapsto x^3 - 3 \cdot x^2$ at $x = 2$. For which values of $x$ is this tangent horizontal?

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**Exercise 28**

(1) Find the slope of the curve given by $y = 5x^3 - 25x^2$ at $x = 3.5$.

Equivalent statement: compute $f'(x)\Big|_{x=3.5}$

(2) Find the equation of the line tangent to the curve at $x = 1$.

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**Exercise 29**

(1) For $f : x \mapsto x^2 + 5$ and the point $A\langle 0; 0\rangle$, what is the equation of the line passing through A, and tangent to $f$?

(2) If $g : x \mapsto ax^2 + b$, what values must $a$ and $b$ take to make $g(x)$ tangent to $t : x \mapsto 3x - 2$ at $x = 5$? What are the coordinates of the contact point?

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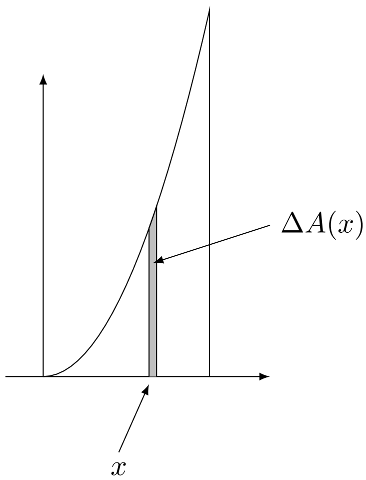
## Area under the curve of $x \mapsto x^2$

**Exercise 30**

To find the area under $f : x \mapsto x^2$ between $x = 0$ and $x = 2$, the idea is to consider the *variation* of the area in order to find the area itself.

Assume that the area under $f$, between 0 and $x$ is given by a function $A(x)$. Consider the variation $\Delta A(x)$, for ultrasmall variation of $x$ noted $\Delta x$.




Even though the exact value of $\Delta A(x)$ may not be directly seen, it can be shown to be between two values, $m$ and $M$ calculated by rectangles.

$$m < \Delta A(x) < M$$

- Give a formula for $m$, using $x$ and $f$.
- Give a formula for $M$, using $x$ and $f$.
- Divide all terms by $\Delta x$.
- Show that all resulting quotients are ultraclose.
- Conclude that the area is given by a function which is the derivative of a known function.

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## Antiderivatives

**Definition 9 (Antiderivative)**
*If $f'$ is the derivative function of $f$, then $f$ is the* ***antiderivative*** *function of $f'$.*

**Exercise 31**
The velocity of an object is given by the derivative of its position (variation of position divided by variation of time).

The acceleration is given by the derivative of the velocity (variation of velocity divided by variation of time).

On earth, the acceleration of a falling body is constant (when there is no air friction) and approximately equal to $9.81\frac{m}{s^2}$, written $g$.

(1) Find the formula for the velocity with respect to time.

(2) Given the formula for velocity, find the formula for the position of a falling body with respect to time.

---

**Exercise 32**
Show that if $F$ is an antiderivative of $f$, then for any constant $C$, $F + C$ is also an antiderivative of $f$.

---

**Exercise 33**
Considering previous exercise, reconsider your answers for exercise 31. Think in terms of units to determine what the constants could represent.

---

**Exercise 34**
Find the antiderivatives for the following:

(1) $x \mapsto 3x$

(2) $x \mapsto x^2$

(3) $x \mapsto 5$

(4) $t \mapsto 3t + 5$

(5) $u \mapsto u^2 + 3u + 5$

(6) $v \mapsto v^3$

Check your results by differentiating them.

---

# THINGS TO LOOK OUT FOR

$f'(a)$ is **NOT** equal to $\frac{\Delta f(a)}{\Delta x}$.

The relation is one of ultracloseness.

$$f'(a) \simeq \frac{\Delta f(a)}{\Delta x}$$

**Practice exercise 5** Answer page 27

Calculate the derivative of the following:

(1) $f : x \mapsto 5x^2 - 10x$ at $x = 2$

(2) $g : x \mapsto 5(x - 10)^2$ at $x = 3$

(3) $h : x \mapsto x^4 + x^3 + x^2 + x + 1$ at $x = 1$

(4) $k : x \mapsto 5x^2 + 10$ at $x = 2$

**Practice exercise 6** Answer page 27

Find the derivative of each of the following functions and specify its domain, starting from the definition.

(1) $a : x \mapsto 1$

(2) $b : x \mapsto |x|$

(3) $c : x \mapsto x$

(4) $d : x \mapsto x^2$

(5) $e : x \mapsto |x^2|$

(6) $f : x \mapsto x^3$

(7) $g : x \mapsto |x^3|$

(8) $h : x \mapsto \dfrac{1}{x}$

(9) $i : x \mapsto \dfrac{1}{x^2}$

**Practice exercise 7** Answer page 27

Find the derivative of each of the following functions and specify its domain, using linearity and the results from the previous exercise.

(1) $a : x \mapsto 2x^2 - 4x + 5$

(2) $b : x \mapsto \dfrac{x^3 + 2x}{7}$

(3) $c : x \mapsto 3x^3 - \dfrac{2}{x}$

(4) $d : x \mapsto \dfrac{x^2 - 2x + 5}{x}$

(5) $e : x \mapsto 5x^3 - 7|x| + 8$

**Practice exercise 8** Answer page 28

Find all the antiderivatives of each of the following functions, using linearity and the results from the exercise 1.

(1) $a : x \mapsto 10x$

(2) $b : x \mapsto x^2$

(3) $d : x \mapsto \dfrac{x}{|x|}$

(4) $e : x \mapsto 3x - 4$

(5) $f : x \mapsto x^2 - 2x + 4$

(6) $g : x \mapsto \dfrac{1}{x^2}$

(7) $h : x \mapsto 2x^2 - \dfrac{1}{2x^2}$

**Practice exercise 9** Answer page 28

Let

$$f : x \mapsto \frac{1}{3}x^3 + \frac{7}{2}x^2 + 12x$$

Calculate its derivative, find where the derivative is positive, where it is negative and where it is equal to zero.

Calculate the intercepts of $f$ and sketch the graph of $f$.

**Practice exercise 10** Answer page 29

Consider the functions differentiated above:

(1) $a : x \mapsto 2x^2 - 4x + 5$

(2) $b : x \mapsto \dfrac{x^3 + 2x}{7}$

For $a$, give the equation the line tangent to the curve at $x = -2$

For $b$, give the equation the line tangent to the curve at $x = 1$

## Answers to practice exercises

**Answers to practice exercice 4**, page 19

(1) $f'(x) = 3$

(2) $g'(x) = 4x - 1$

(3) $h'(x) = 15x^2 + 4x - 1$

(4) $k'(x) = 15x^2 + 4x + 3$

**Answers to practice exercice 5**, page 24

(1) $f'(2) = 10$

(2) $g'(3) = -70$

(3) $h'(1) = 10$

(4) $k'(2) = 20$

**Answers to practice exercice 6**, page 24

(1) $a'(x) = 0$ Domain=$\mathbb{R}$

(2) $b'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ Domain=$\mathbb{R} \setminus \{0\}$

(3) $c'(x) = 1$ Domain=$\mathbb{R}$

(4) $d'(x) = 2x$ Domain=$\mathbb{R}$

(5) $e'(x) = 2x$ Domain=$\mathbb{R}$

(6) $f'(x) = 3x^2$ Domain=$\mathbb{R}$

(7) $g'(x) = \begin{cases} 3x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$ Domain=$\mathbb{R}$

(8) $h'(x) = \dfrac{-1}{x^2}$ Domain=$\mathbb{R}$

(9) $i'(x) = \dfrac{-2}{x^3}$ Domain=$\mathbb{R}$

**Answers to practice exercice 7**, page 24

(1) $a'(x) = 4x - 4$ Domain=$\mathbb{R}$

(2) $b'(x) = \dfrac{3x^2 + 2}{7}$ Domain=$\mathbb{R}$

(3) $c'(x) = 9x^2 + \frac{2}{x^2}$ Domain=$\mathbb{R} \setminus \{0\}$

(4) $d'(x) = 1 - \frac{5}{x^2}$ Domain=$\mathbb{R} \setminus \{0\}$

(5) $e'(x) = \begin{cases} 15x^2 - 7 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ 15x^2 + 7 & \text{if } x < 0 \end{cases}$ Domain=$\mathbb{R} \setminus \{0\}$

**Answers to practice exercice 8**, page 25

(1) $A(x) = 5x^2 + C$ for any $C \in \mathbb{R}$

(2) $B(x) = \frac{x^3}{3} + C$ for any $C \in \mathbb{R}$

(3) $D(x) = C$ for any $C \in \mathbb{R}$ (function undefined at $x = 0$)

(4) $E(x) = \frac{3}{2}x^2 - 4x + C$ for any $C \in \mathbb{R}$

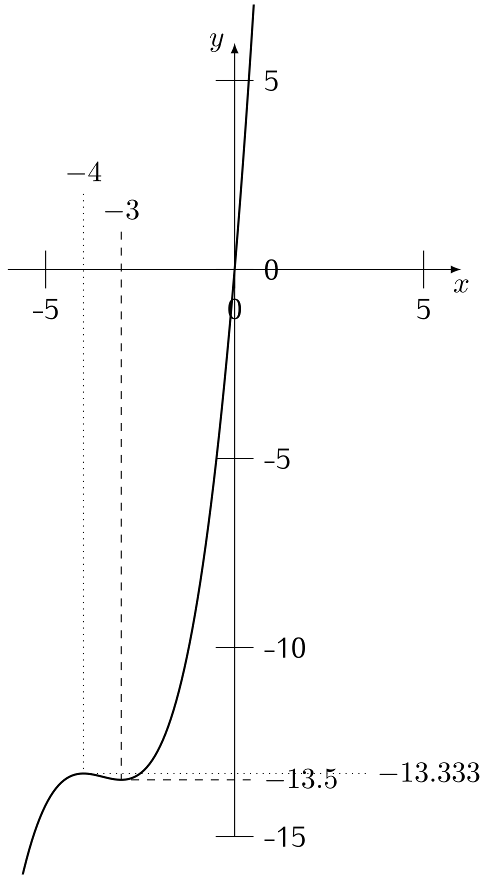
(5) $F(x) = \frac{x^3}{3} - x^2 + 4x + C$ for any $C \in \mathbb{R}$

(6) $G(x) = -\frac{1}{x} + C$ for any $C \in \mathbb{R}$

(7) $H(x) = \frac{2}{3}x^3 + \frac{1}{2x} + C$ for any $C \in \mathbb{R}$

**Answers to practice exercice 9**, page 25

$f(x) = x\left(\frac{1}{3}x^2 + \frac{7}{2}x + 12\right)$
$\mathcal{S} = \{0\}$
$f'(x) = x^2 + 7x + 12 = (x+3)(x+4)$
$\mathcal{S}' = \{-3, -4\}$




**Answers to practice exercice 10**, page 25

(1) $t_a : x \mapsto -12x - 3$

(2) $t_b : x \mapsto \dfrac{5}{7}x - \dfrac{2}{7}$

# 4
# Continuity

Informally: a function is continuous at $x = a$ if it is where you would expect it to be by observing where it is in the neighbourhood of $a$.

**Definition 10 (Continuity )**
*Let $f$ be a real function defined around $a$. We say that $f$ **is continuous at** $a$ if (for any $x$)*

$$x \simeq a \Rightarrow f(x) \simeq f(a).$$

The continuity of $f$ at $a$ is a property of $f$ and $a$. Hence the context is given by $f$ and $a$.
The definition of continuity can also be interpreted in the following ways:

**Definition 11 (Continuity: equivalent definition)**
*Let $f$ be a real function defined around $a$. We say that $f$ **is continuous at** $a$ if*

$$f(a + \Delta x) \simeq f(a) \text{ not depending on } \Delta x.$$

(As for the derivative, the context is $f$ and $a$.)

**Exercise 35**
Show that $f : x \mapsto x^3$ is continuous at $a = 2$.

---

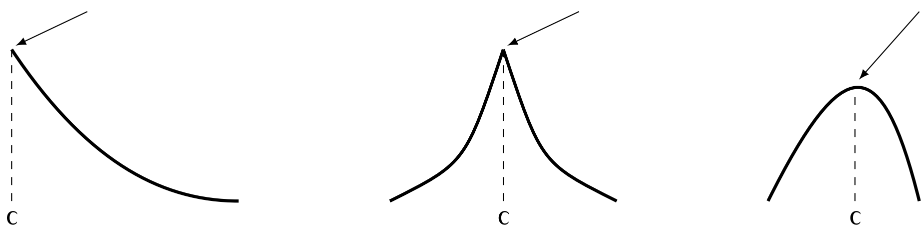
**Theorem 8 (Critical Point Theorem)**
*Let $f$ be a continuous function on $I$ and suppose that $c$ is a point in $I$ and $f$ has either a maximum or a minimum at $c$. Then one of the following three things must happen:*

*(1) $c$ is an end point of $I$.*

*(2) $f'(c)$ is undefined.*

*(3) $f'(c) = 0$*

The critical point theorem graphically:




The two first cases are direct observation. The third case id theorem 6.

**Exercise 36**

Show whether $f : x \mapsto \dfrac{x}{x^2+1}$ is continuous for all values of $x$.

exo

(1) Show that $f : x \mapsto |x|$ is continuous at $x = 0$, at $x = 1$, at $x = -1$ and at $x$ in general.

(2) Show that $g : x \mapsto \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$ and at $x$ in general.

(3) Show that $g : x \mapsto \begin{cases} x^2 & \text{if } x \geq -1 \\ x^3 & \text{if } x < -1 \end{cases}$ is not continuous at $x = -1$ but is continuous for all other values of $x$.

**Exercise 37**

Prove the following theorem:

**Theorem 9**

*If a real function $f$ is differentiable at $a$ then $f$ is continuous at $a$.*

(1) Give a direct proof.

(2) Give a proof by contrapositive.

**Exercise 38**

Use an induction proof to show that $x \mapsto x^n$ is continuous for all $n$.

**Exercise 39**

Prove the following theorem:

**Theorem 10**

*Let $f$ and $g$ be two real functions continuous at $a$. Then*

*(1) $f \pm g$ is continuous at $a$.*

*(2) $f \cdot g$ is continuous at $a$.*

*(3) $\dfrac{f}{g}$ is continuous at $a$ if $g(a) \neq 0$.*

**Exercise 40**

Prove the following theorem:

**Theorem 11**

*Let $f$ and $g$ be two real functions. If $f$ is continuous at $a$ and $g$ is continuous at $f(a)$, then $g \circ f$ is continuous at $a$.*

**Exercise 41**

Use an induction proof to show that $x \mapsto a_0 + \sum_{k=1}^{n} a_k x^k$ is continuous for all $n$.

---

**Definition 12 (Continuity on an Interval)**

*(1) Let $f$ be a real function defined on the open interval $]a;b[$. Then $f$ is **continuous on** $]a;b[$ if $f$ is continuous at all $x \in ]a;b[$.*

*(2) Let $f$ be a real function defined on the closed interval $[a;b]$. Then $f$ is **continuous on** $[a;b]$ if $f$ is continuous at all $x \in ]a;b[$ and if $f$ continuous on the right at $a$ and on the left at $b$.*

Informally: a function is continuous on an interval if its curve can be drawn without lifting the pencil, or if the function is where you expect it to be if it is hidden by a vertical line.

**Exercise 42**

Determine whether $f : x \mapsto x^2$ is continuous on its domain.

---

Clearly, if $f$ and $g$ are continuous on an interval $I$ then the sum, difference, product and quotient (if $g(x) \neq 0$) are continuous on $I$. Moreover, if $g$ is continuous on an interval containing $f(I)$ then $g \circ f$ is continuous on $I$.

**Exercise 43**

Show whether the following functions are continuous on the given intervals.

(1) $f_1 : x \mapsto \frac{1}{3}x + \sqrt{2}$ on $\mathbb{R}$

(2) $f_2 : x \mapsto x^2 - 3x - 1$ on $\mathbb{R}$

(3) $f_3 : x \mapsto \dfrac{x+2}{x-1}$ on $]1; +\infty[$

---

**Exercise 44**

Determine whether $f : x \mapsto \frac{1}{x}$ is continuous on its domain.

---

**Theorem 12 (Intermediate Value theorem)**

*Let $f$ be a real function continuous on $[a;b]$. Let $d$ be a real number between $f(a)$ and $f(b)$. Then there exists $c$ in $[a;b]$ such that $f(c) = d$.*

This theorem does not tell us how to find the root or the value $c$ such that $f(c) = d$. It only asserts the *existence* of such a number. For specific functions where we can calculate the roots explicitly this theorem is not really necessary but, when proving theorems about continuous functions in general, it is the only way to know that there is a root.

**Exercise 45**

Give an example of a function $f$ discontinuous on $[a;b]$ with $f(a) < 0$ and $f(b) > 0$ such that there is no $c$ in the interval $[a;b]$ such that $f(c) = 0$.

---

**Exercise 46**

Proving theorem 12.

Let $f$ be continuous on an interval $[a;b]$.

Assume $d = 0$ and $f(a) < 0 < f(b)$.

The context is $f$, $a$, $b$ and 0. Take an ultralarge positive integer $N$ and partition $[a;b]$ into $N$ even parts, each of ultrasmall length $\Delta x = \frac{b-a}{N}$. We thus have $x_0 = a,\ x_1 = x_0+\Delta x,\ \ldots,\ x_N = b$.

Call $x_j$ the first point of the partition such that $f(x_j) \geq 0$. Hence $f(x_{j-1}) < 0$.

(1) Let $c$ be the observable part of $x_j$. Is it the observable part of $x_{j-1}$?

(2) Is $f(c)$ observable?

(3) How close are $f(x_j)$ and $f(x_{j-1})$?

(4) How close is $f(c)$ from $f(x_j)$ and $f(x_{j-1})$?

(5) What is the value of $f(c)$?

(For $d \neq 0$ the theorem would hold for $g(x) = f(x)+d$; for $f(a) > f(b)$, reverse all inequality symbols.)

---

**Definition 13**

*A function has **maximum** (respectively **minimum**) on an interval $I$ if there is a $c \in I$ such that for any $x \in I$ we have $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$).*
*If a point is either a maximum or a minimum, it is an **extremum**.*

**Theorem 13 (Extreme value)**

*Let $f$ be a continuous function on $[a;b]$. Then it has a maximum and a minimum on $[a;b]$.*

**Exercise 47**

Without loss of generality, we consider the case of a maximum (for the minimum replace $f$ by $-f$). Context is $f, a$ and $b$.

We proceed similarly to exercise 46.

Let $f$ be continuous on an interval $[a;b]$.

Take an ultralarge positive integer $N$ and partition $[a;b]$ into $N$ even parts, each of length $\Delta x = \frac{b-a}{N}$. We thus have $x_0 = a,\ x_1 = x_0 + \Delta x,\ \ldots,\ x_N = b$.

Call $x_j$ the first point of the partition such that $f(x_j) \geq f(x_i)$ for any $i$ between 0 and $N$.

(1) Call $c$ the observable part of $x_j$. Is $f(c)$ observable?

(2) Let $x$ be observable. Then there is an $i$ such that $x_i \leq x \leq x_{i+1}$. Using continuity, conclude that $f(x) \leq f(x_j)$.

(3) By the closure principle, conclude that $f(c)$ is the maximum.

---

## Continuity and Differentiability

**Theorem 14 (Rolle)**
*Let $f$ be a real function continuous on $[a;b]$ and differentiable on $]a;b[$. If $f(a) = f(b)$, then there is a $c \in ]a;b[$ such that*

$$f'(c) = 0.$$

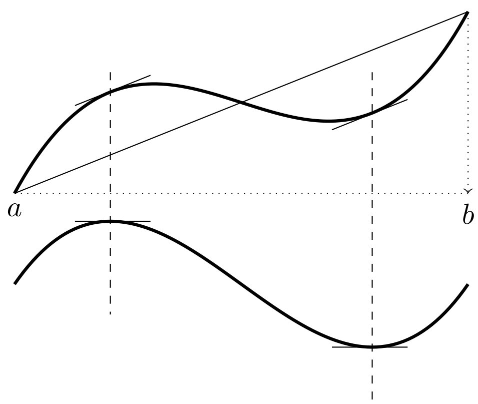
**Exercise 48**
Prove Rolle's theorem.

---

**Theorem 15 (Mean Value)**
*Let $f$ be a real function continuous on $[a;b]$ and differentiable on $]a;b[$. Then there is a $c \in ]a;b[$ such that*

$$f(b) - f(a) = f'(c) \cdot (b - a).$$

**Exercise 49**
Consider $g$ which is obtained by subtracting the line $\ell(x)$ through $(a, f(a))$ and $(b, f(b))$ from the function $f$ i.e., $g(x) = f(x) - \ell(x)$.




Show that $g$ satisfies Rolle's theorem and conclude with the mean value theorem.

---

**Exercise 50**
Let $f$ be continuous and positive on $[a;b]$
Assuming the area function under $f$ is given by $A$. Show how $A$ can be bounded above and below. Show that there is a value $c \in [a;b]$ such that $A = f(c) \cdot (b - a)$.

---

**Exercise 51**

Prove the following theorem:

**Theorem 16**

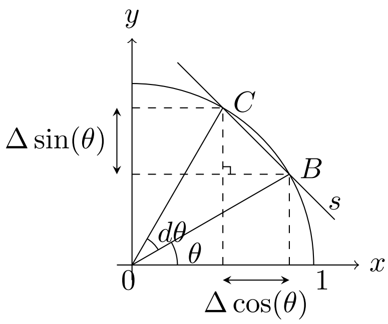
*The antiderivative of a function – when it exists – is unique up to an additive constant i.e., for any constant $C$*

$$f' = g' \Rightarrow f = g + C$$

---

**Exercise 52**

Consider the trigonometric circle. The chord $BC$ is shorter than the arc $BC$.




Show that sine and cosine are continuous functions.

---

## Optimisation Problems

**Exercise 53**

A $1l$ milk pack is made of cardboard. Its sides can only be rectangles. The height is twice one of the other two dimensions. The area of the pack must be minimal.

What are the dimensions of the pack?

---

**Exercise 54**

Imagine you want to protect a part of a rectangular garden against a long wall. You have 100m of fence. (No fence is needed against the wall.)

What is the biggest area that you can protect?

---

**Exercise 55**

A cylindrical jar has a volume defined by its radius and its height. If it contains one litre (1dm$^3$), what are the dimensions that will make it have the least area?

---

**Exercise 56**

Find the length and width of the rectangle inscribed within the ellipse given by the formula $4x^2 + y^2 = 16$ (sides parallel to the coordinate axes) such that its area is maximal.

---

**Exercise 57**

Let $\mathcal{P}$ be the parabola given by $x \mapsto x^2$ and $A$ be the point $\langle 0; 5 \rangle$.

Find the point(s) on the parabola $\mathcal{P}$ such that its (their) distance to $A$ is minimal.

---

## Bending

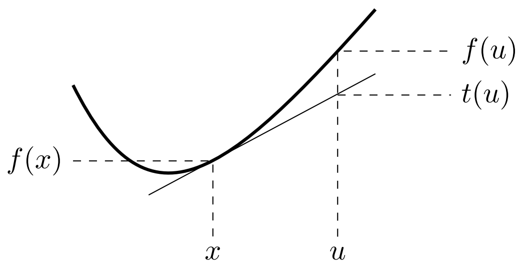
**Definition 14 (Second Derivative)**
*Let $f$ be a function differentiable at $a$. If $f'$ is also differentiable at $a$, then we say that $f$ is differentiable twice at $a$ and $(f')'(a) = f''(a)$*[a]

[a]pronounced: "eff double prime"

**Definition 15**
*Let $f$ be differentiable on an interval $I$. The curve of $f$ is* ***bending upwards on*** *$I$ if for every $x, u \in I$, $f(u)$ is above the line tangent to $f$ at $(x, f(x))$, i.e.,*

$$f(u) \geq f'(x)(u - x) + f(x).$$

*The curve of $f$ is* ***bending downwards on*** *$I$ if $(-f)$ is bending upwards.*




For ultrasmall $(u - x)$ this can be rephrased in the following manner:

**Definition 16**
*A differentiable function $f$ is bending upwards at $a$ if*

$$f(a + \Delta x) \geq f(a) + f'(a) \cdot \Delta x.$$

**Theorem 17 (Bending and Second Derivative)**
*Let $f$ be twice differentiable on an interval $I$. Then*

*(1) If $f''(x) \geq 0$ whenever $x \in I$ then $f$ is bending upwards on $I$.*

*(2) If $f''(x) \leq 0$ whenever $x \in I$ then $f$ is bending downwards on $I$.*

**Exercise 58**
Use the mean value theorem to prove theorem 17.

# 5

# Differential Calculus

For the following rules, the proofs proceed by steps:

(1) Definition of the derivative.

(2) Definition of $\Delta$.

(3) Definition of operations on functions.

(4) Expansion of $f(a + dx)$ as $f(a) + \Delta f(a)$.

(5) Division by $dx$.

(6) Algebra.

(7) Definition of the antiderivative for the inverse rule about integration.

**Exercise 59**

Explain why if $f$ is differentiable at $a$, then $\Delta f(a) \simeq 0$.

---

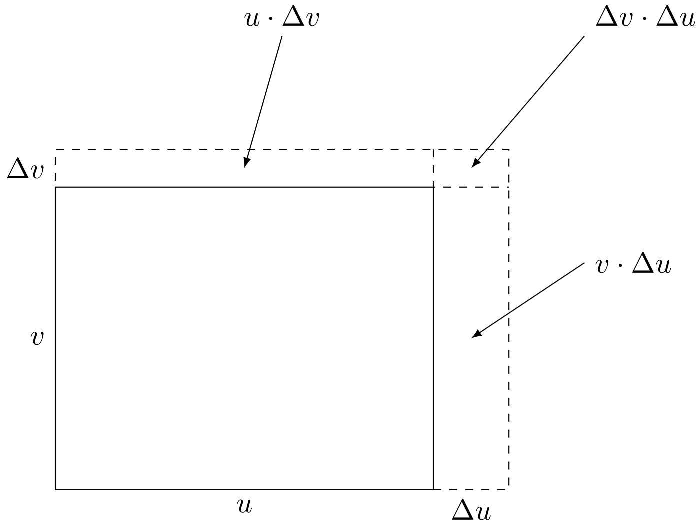
The previous property can be rewritten using the $y = f(x)$ notation, where $y$ is a dependent variable. Then if $y'$ exists, we have $y' \simeq \frac{\Delta y}{\Delta x}$ and $\Delta y \simeq 0$.

## Product

When two different functions are involved, it is common practice to write $f(x) = u$ and $g(x) = v$ then $\Delta f(x) = \Delta u$ and $\Delta g(x) = \Delta v$.

Consider the product $u \cdot v$ and its variation (a product $a \cdot b$ can be interpreted as the area of a rectangle with sides $a$ and $b$).

When $x$ varies to $x + \Delta x$, $u$ varies to $u + \Delta u$ and $v$ varies to $v + \Delta v$.




Then $u \cdot v$ varies to $v \cdot u + v \cdot \Delta u + \Delta v \cdot u + \Delta v \cdot \Delta u$ hence

$$\Delta(u \cdot v) = v \cdot \Delta u + \Delta v \cdot u + \Delta v \cdot \Delta u$$

**Exercise 60**
Divide the expression above by $\Delta x$ and justify that $\frac{\Delta u \cdot \Delta v}{\Delta x} \simeq 0$ to prove

**Theorem 18**
*Let $u$ and $v$ be two differentiable functions, then*

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

---

This theorem can also be written:

Let $f$ and $g$ be two real functions differentiable at $a$. Then the function $f \cdot g$ is differentiable at $a$ and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

**Exercise 61**
Using the derivatives of $f : x \mapsto x^2$ and $g : x \mapsto x^3$, calculate the derivative of $h : x \mapsto x^5$ $(= x^2 \cdot x^3)$.

---

**Exercise 62**
Prove :

**Theorem 19**

$$(x^n)' = n \cdot x^{n-1}$$

by induction.

---

## Induction

If

(1) The property holds for $n = 0$ (or $n = 1$),

(2) Assuming the property holds for $n$ greater than 0 (or 1), we can prove that it also holds for $n + 1$,

then the property holds for all $n$.

A proof that this method of proof is valid can be given by contradiction. Assume (1) and (2) have been checked but that there is a value $m$ such that the property does not hold for $m$. Then $m > 1$ since that has been proven to be true. Let $n$ be the smallest number such that the property does not hold. (This number is not zero because of (1).) Then the property holds for $n - 1$. But by (2), this proves that the property holds for $n$: a contradiction. So there can be no number for which the property does not hold.

**Exercise 63**

Similar to exercise 30: Calculate the area between $y = 5x^4 - 3x^3 + 2x^2 - 10$ and the $x$-axis from $x = -1$ to $x = 1$.

---

**Exercise 64**

Sketch the curve of $f : x \mapsto x^2$ and $g : x \mapsto x^3$. Calculate the points where $f(x) = g(x)$

Calculate the closed geometric area of the surface between the two curves.

---

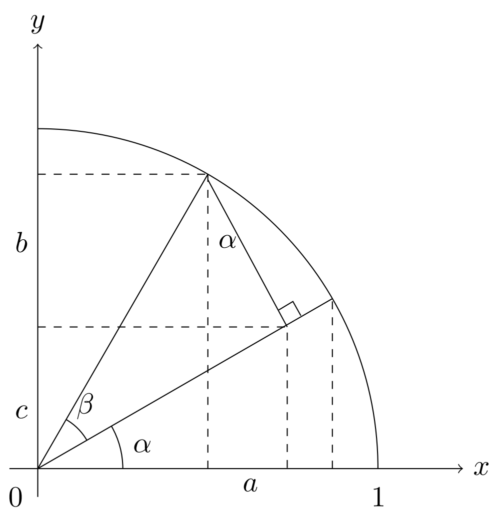
## Circular functions

**Exercise 65**

Observe the following drawing where the angle $\beta$ has been drawn on top of the angle $\alpha$.

(1) Explain why the angle right at the top is equal to $\alpha$

(2) Express the lengths of $a$, $b$ and $c$ in terms of $\sin(\alpha), \cos(\alpha), \sin(\beta)$ and $\cos(\beta)$.



**Exercise 66**
Finish the proof of

**Theorem 20**

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

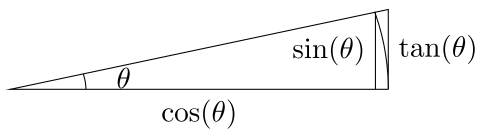
**Exercise 67**
Use the definition of the derivative and theorem 20 to expand $\Delta\sin(a)$

**Exercise 68**
To continue, you will need to prove theorem 21:

**Theorem 21**

$$\frac{\sin(\Delta\theta)}{\Delta\theta} \simeq 1.$$

Suppose first that $\theta > 0$ is in the first quadrant.




Comparing the area of the sector with that of the inside and outside triangles, we obtain

inside triangle $\leq$ sector $\leq$ outside triangle.

Rewrite this chain of inequalities replacing the areas by the corresponding formulae.

By using $-\theta$ if $\theta$ is negative, we see that the same inequalities are true for negative $\theta$ (in the fourth quadrant).

Let $\theta$ be ultrasmall. By continuity, $\cos(\theta) \simeq 1$. Then conclude the proof of the theorem.

**Exercise 69**
Show that

$$\frac{1-\cos(\Delta\theta)}{\Delta\theta} \simeq 0.$$

Hint: multiply above and below by $(1+\cos(\Delta\theta))$

**Exercise 70**

Using theorem 21 and previous exercise, find the derivative of $\sin(x)$ and of $\cos(x)$.

---

These results are summarised here:

**Theorem 22**

*(1)* $\sin'(\theta) = \cos(\theta)$

*(2)* $\cos'(\theta) = -\sin(\theta)$

**Exercise 71**

Let $c$ be a constant, considered as a constant function. What is $\Delta c$? and use this to conclude that

**Theorem 23**

*Let $c$ be a constant. Then*

$$c' = 0$$

---

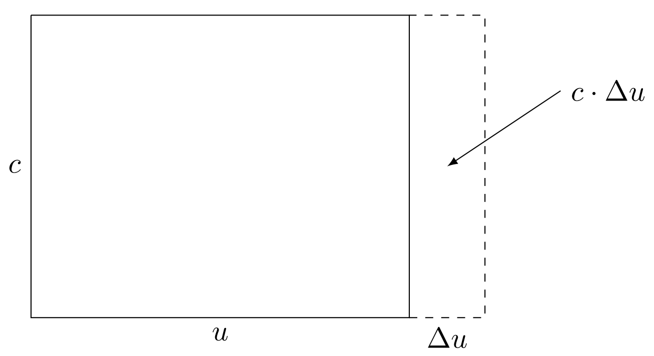
This theorem can also be written:

Let $c \in \mathbb{R}$ and $f : x \mapsto c$, for $x \in \mathbb{R}$

$$f'(x) = 0.$$

Consider the product $c \cdot u$ for constant $c$ and differentiable function $u$, then when $x$ varies to $x + \Delta x$ the product $c \cdot u$ varies $to c \cdot u$ to $c \cdot u + c \cdot \Delta u$, hence

$$\Delta(c \cdot u) = c \cdot \Delta u$$



**Exercise 72**

Divide the expression above by $\Delta x$ to prove

**Theorem 24**

*Let $c$ be a constant and $u$ a differentiable function. Then*

$$(c \cdot u)' = c \cdot u'$$

---

This theorem can also be written:

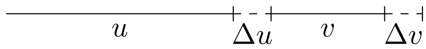
Let $c \in \mathbb{R}$ and $f$ be a real function differentiable at $a$. Then the function $c \cdot f$ is differentiable at $a$ and

$$(c \cdot f)'(a) = c \cdot f'(a).$$

A function such as $f : x \mapsto (x^3+2x)^4$ can be decomposed as a composition of $f_1 : x \mapsto x^3+2x$ and $f_2 : x \mapsto x^4$. Then $f = f_2 \circ f_1$.

## Sum and Difference

Consider the sum. When $x$ varies to $x + \Delta x$, $u$ varies to $u + \Delta u$ and $v$ varies to $v + \Delta v$.




Then

$$\Delta(u + v) = \Delta u + \Delta v$$

**Exercise 73**

Divide the expression above to prove:

**Theorem 25**

*Let $u$ and $v$ be differentiable functions. Then*

$$(u + v)' = u' + v'$$

---

This theorem can also be written:

Let $f$ and $g$ be real functions differentiable at $a$. Then the function $f + g$ is differentiable at $a$ and

$$(f + g)'(a) = f'(a) + g'(a).$$

**Exercise 74**

Find the derivatives of $h : x \mapsto x^3 + x^2$ and $k : x \mapsto 5x^3 - 7x^2$.

---

## Composition

**Theorem 26 (Chain Rulle)**

*Let $u$ by a differentiable function of $v$ and $v$ a differentiable function of $x$. Then*

$$(u \circ v)' = u' \cdot v'$$

**Exercise 75**

Prove the chain rule.

---

**Exercise 76**

Prove that this formula holds also if $\Delta v = 0$.

---

This theorem can also be written:

Let $f$ and $g$ be real functions such that $g$ is differentiable at $a$ and $f$ is differentiable at $g(a)$. The the function $f \circ g$ is differentiable at $a$ and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

**Exercise 77**

Give the derivatives of the following functions:

(1) $f : x \mapsto (x^3 + 2x)^4$

(2) $g : x \mapsto (5x^3 + 3x^2)^{13}$

---

**Exercise 78**

Use $(\sqrt{x})^2 = x$ and theorem 26 to find the derivative of $y = \sqrt{x}$ (for $x > 0$) – assuming it exists.

---

**Exercise 79**

Give the derivatives of the following functions:

(1) $f : x \mapsto (\sqrt{x} + 1)^4$

(2) $g : x \mapsto \sqrt{5x^3 + 3x^2}$

(3) $h : x \mapsto \sqrt{x^2}$

---

**Exercise 80**

Find the derivatives of the following:

(1) $y = \sqrt{3x^3 + 2x + 1}$

(2) $y = (x^2 + 3)^5$

(3) $y = (ax + b)^n$

(4) $y = \sqrt{x^3 + 1}$

---

**Exercise 81**

Use the definition of the derivative to find $f'(x)$ for $f : x \mapsto \frac{1}{x}$

---

**Exercise 82**

Use the previous exercise and the chain rule to find the derivative of $\frac{1}{f(x)}$ assuming $f(x) \neq 0$ and $f'(x)$ exists.

---

## Quotient

**Exercise 83**

Use all previous results to prove:

**Theorem 27**

*Let $u$ and $v$ be differentiable functions with $v \neq 0$, then*

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

Also written:

Let $f$ and $g$ be two real functions differentiable at $a$ and $g(a) \neq 0$. Then the function $\dfrac{f}{g}$ is differentiable at $a$ and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g^2(a)}.$$

---

**Exercise 84**

Calculate $\tan'(x)$ using $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

---

**Exercise 85**

Find the slope of $f : x \mapsto \dfrac{x^2 - 2x + 1}{x^3 + x^2}$ at $x = 1$.

---

**Exercise 86**

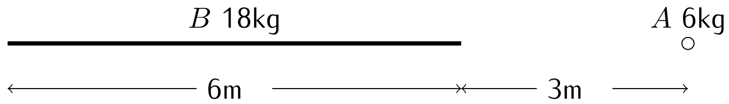
Show that for $m \in \mathbb{Z}$

$$(x^m)' = m \cdot x^{m-1}.$$

---

**Exercise 87**

Given that the gravitational force between two masses is $F = G\dfrac{m_1 \cdot m_2}{d^2}$ (where $d$ is the distance between the two masses and $G$ the universal constant of gravitation), what is the force between objects $A$ and $B$ in the following situation? (For simplicity, the linear mass will be considered to have no width and the other will be considered reduced to a point.)




**Practice exercise 11** Answer page 75

Differentiate the following for general $x$:

(1) $f : x \mapsto 5x^4 + x^3 - 2x^2 + 25$

(2) $g : x \mapsto 5\sqrt{3}\, x^2 - 100$

(3) $h : x \mapsto \dfrac{x^2 + 2x - 1}{x^3 - 5}$

(4) $j : x \mapsto 5x^4 + \dfrac{1}{3x^2 - 2x + \pi}$

(5) $k : x \mapsto (5x + 2) \cdot \dfrac{1}{5x + 2}$

(6) $l : x \mapsto \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4}$

(7) $m : x \mapsto \dfrac{1 + x}{1 + \frac{1+x}{x^2}}$

**Practice exercise 12** Answer page 75

Sketch the curve of $y = -(x - 3)(x + 1)(x - 1)$.

**Practice exercise 13** Answer page 75

Let $y = \dfrac{10x}{x^2 + 1}$. Sketch the curve and give the equation of the line tangent to the curve at $x = 3$.

**Practice exercise 14** Answer page 76

Consider each of the following as a function $f$, find the corresponding derivative function $f'$.

(1) $x^3 + x^2 + 2x - 4$

(2) $-x^3 + 2x^2 - 2x + 1$

(3) $\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$

(4) $\frac{1}{3}(x - 2)^3$

(5) $\dfrac{x^2}{x + 2}$

(6) $x - 1 + \dfrac{9}{x + 1}$

(7) $\dfrac{4x^2 + 4x + 5}{4x + 2}$

(8) $\dfrac{-x^2 - 2x - 1}{x + 3}$

(9) $|x - 2|$

(10) $\dfrac{x^2}{|x| + 2}$

(11) $x + 2 - \dfrac{1}{x + 1}$

(12) $|x^3 - 6x^2 + 11x - 6|$

**Exercise 88**

Find the derivative of the following functions. Since they are piecewise defined, the answer will be in 3 parts – one special point is the meeting point for both rules.

(1)

$$f : x \mapsto \begin{cases} x^2 & \text{if } x \geq 1 \\ 2x - 1 & \text{if } x < 1 \end{cases}$$

(2)

$$g : x \mapsto \begin{cases} x^2 & \text{if } x > 2 \\ x + 2 & \text{if } x \leq 2 \end{cases}$$

(3)

$$h : x \mapsto \begin{cases} x^2 & \text{if } x \geq 3 \\ 2x & \text{if } x < 3 \end{cases}$$

---

**Practice exercise 15** Answer page 76

Find the derivatives of the following:

(1) $f_1 : x \mapsto \sqrt{3x^3 + 2x + 1}$

(2) $f_2 : x \mapsto (x^2 + 3)^5$

(3) $f_3 : x \mapsto (ax + b)^n$

(4) $f_4 : x \mapsto \sqrt{x^3 + 1}$

(5) $f_5 : x \mapsto \sin(x^2 + 3x)$

(6) $f_6 : \theta \mapsto \cos^2(3\theta)$

(7) $f_7 : u \mapsto \sin(\sin(u))$

(8) $f_8 : x \mapsto \tan^2(\tan^2(x^2))$

(9) $f_9 : v \mapsto \dfrac{\sin(v)}{\tan(v)}$

(10) $f_{10} : x \mapsto \sin^2(x) + \cos^2(x)$

## The differential

It is traditional to ue $dx$ for ultrasmall $\Delta x$.

**Definition 17**

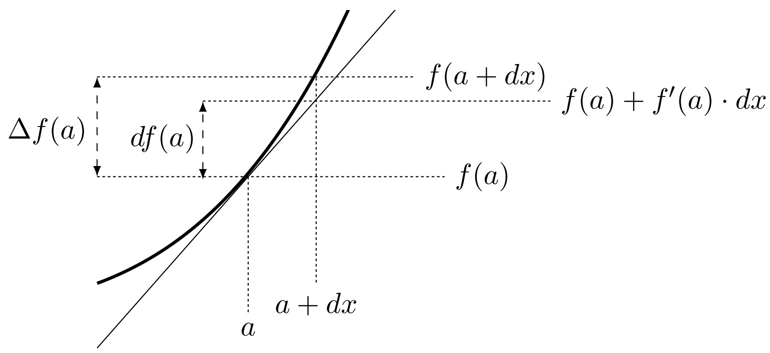
*Let $f$ be a real function differentiable on an interval around $a$. Let $\Delta x$ be ultrasmall. The **differential of $f$ at** $a$, written $df(a)$, is*

$$df(a) = f'(a) \cdot dx.$$

⚠ While we write $dx = \Delta x$, we cannot write $dy = \Delta y$. We have $\Delta y = y' + \varepsilon \cdot dx$.

Thus
$$\frac{df(a)}{dx} = f'(a)$$
or still (if we use $y = f(a)$)
$$\frac{dy}{dx} = y'$$
.

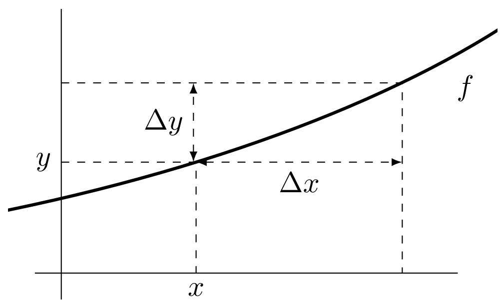
If $f$ is differentiable the following holds:
$$\frac{\Delta f(a)}{\Delta x} \simeq \frac{df(a)}{dx}$$
Whereas $\Delta f(a)$ is the variation of the function, the differential $df(a)$ is the variation along the tangent line.

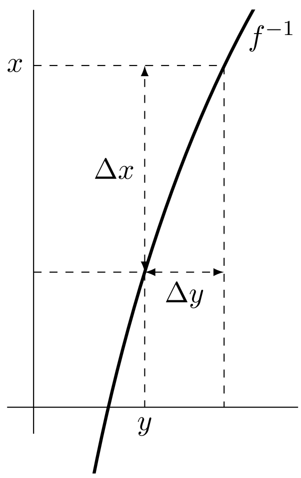



Let $f$ be a function. Recall that the inverse function of $f$, if it exists, is written $f^{-1}$ and is such that $f^{-1}(f(x)) = x$ amd if we write $f(x) = y$ then we also have $f(f^{-1}(y)) = y$.

⚠ $f^{-1}(x)$ is <u>not</u> $\frac{1}{f(x)}$.

A function has an inverse if the image of its curve by a symmetry through the $y = x$ axis is the curve of a function.



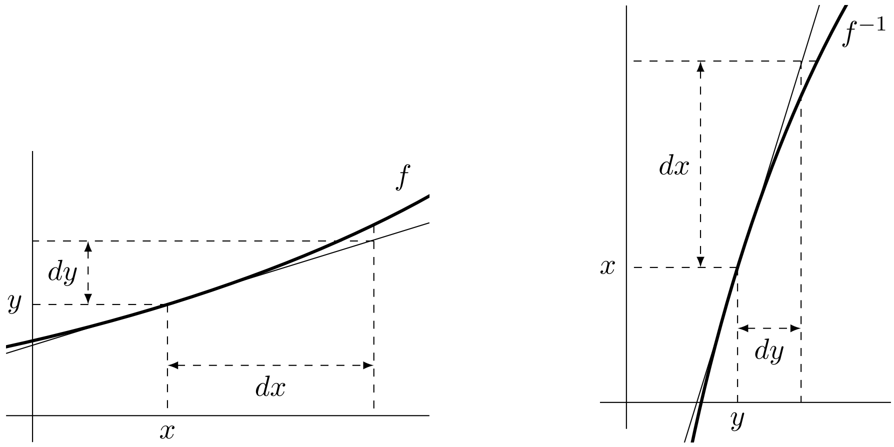
**Theorem 28 (Derivative of the Inverse)**
*If $f : I \to J$ is a function, differentiable on $I$ and has an inverse $f^{-1}$, and $f'(a) \neq 0$ then this inverse is differentiable at $b = f(a) \in J$ and*

$$\frac{df^{-1}(b)}{dy} = \frac{1}{f'(a)}.$$

This can also be written:

$$\frac{dx}{dy} = \frac{1}{y'}$$

You may also use the following drawing to observe that the slope of the tangent of the inverse is the reciprocal of the slope of the original tangent.




**Exercise 89**
Find the derivative of $y = x^{\frac{1}{n}}$.

---

**Exercise 90**
Find the derivative of $y = x^{\frac{m}{n}}$.

---

This shows that the rule in exercise 62 holds also for rational $n$.

**Exercise 91**
Use $\mid x \mid = \sqrt{x^2}$ to find an expression for the derivative of $\mid x \mid$.

---

**Exercise 92**
**Difficult exercise!**
Let $h$ be ultrasmall relative to 1.

$$H : x \mapsto \begin{cases} 0 & \text{if } x \leq -h \\ \frac{1}{2h}(x+h) & \text{if } -h < x < h \\ 1 & \text{if } x \geq h \end{cases}$$

(1) What is the context of the function?

(2) Calculate $H'(x)$.

(3) Sketch $H$, first with horizontal scale $[-2;2]$ and vertical scale $[0;1]$ then, for same vertical scale, take a horizontal scale $[-2\cdot h; 2\cdot h]$.

---

**Exercise 93**

For the inverse functions, it is convenient to use the differential.
Prove the following theorem:
Hint: Suppose that $\arcsin(x) = y$ i.e., $\sin(y) = x$. Then $\arcsin'(x) = \frac{dy}{dx} = \frac{dy}{d\sin(x)}$.

**Theorem 29**

*(1)* $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$

*(2)* $\arccos'(x) = -\dfrac{1}{\sqrt{1-x^2}}$

*(3)* $\arctan'(x) = \dfrac{1}{1+x^2}$

---

**Exercise 94**

Let $\varepsilon$ be ultrasmall relative to 1. Consider the function

$$H : x \mapsto \frac{1}{2} + \frac{1}{\pi}\cdot \arctan\left(\frac{x}{\varepsilon}\right).$$

Calculate the value of $H$ at nonzero observable values, at zero.
Calculate $H'(x)$ and sketch the curves of $H$ and $H'$.
Calculate the value of $H'$ at nonzero observable values, at zero.

---

**Exercise 95**

(1) Show that $x \mapsto \cos\left(\frac{1}{x}\right)$ cannot be extended continuously at $x = 0$.

(2) Show that

$$x \mapsto \begin{cases} x^2\cdot\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$ but that its derivative $x \mapsto g'(x)$ is not continuous at $0$.

---

**Exercise 96**

Compute the derivatives of the following:

(1) $f : x \mapsto \sin^2(3x + \pi)$

(2) $g : x \mapsto x \cdot \sin(x^2 + 1)$

(3) $h : x \mapsto \sin^2\left(\frac{x}{x^2+1}\right) + \cos^2\left(\frac{x}{x^2+1}\right)$

(4) $j : x \mapsto 1 + \tan^2(x)$

---

**Exercise 97**

(1) Show that $f : x \mapsto \sin^6(x) + \cos^6(x) + 3\sin^2(x)\cos^2(x)$ is a constant function. (Hint: use the derivative...)

(2) At what values does $f : x \mapsto \sin(x) + \cos(x)$ have stationary points?

(3) What is the equation of the straight line tangent to $y = \sin^2(x)$ at $x = \frac{\pi}{4}$?

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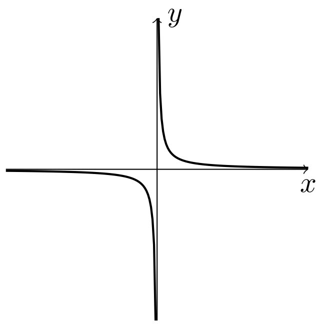
# 6
# Asymptotes

**Exercise 98**

Consider the function $f : x \mapsto \frac{1}{x}$.




(1) What is the domain of this function? Specify the context.

(2) What happens to the curve close to the vertical axis i.e., for values of $x$ close to 0? Consider ultrasmall values of $x$.

(3) What happens to the curve close to the horizontal axis? i.e., for very large values of $x$? Consider ultralarge values of $x$ (positive or negative).

(4) Draw this function for a horizontal range of $[-100; 100]$ and a vertical range of $[-100; 100]$.

(5) Does $f$ have a limit at 0?

---

Informally: For a given function $f$, a straight line is **an asymptote** of the function $f$ if it is ultraclose to the function when either

- $x$ tends to $\pm\infty$ (horizontal or oblique asymptote).
- $y$ (or $f(x)$) tends to $\pm\infty$ (vertical asymptote).

**Definition 18**
*A real function $f$ **has a vertical asymptote at** $x = a$ if $f(x)$ is positive or negative ultralarge for $x \simeq a$, $x$ being less than $a$ or $x$ being greater than $a$.*
*If it is the case for $x$ greater than $a$, we write*

$$x \simeq a_+ \Rightarrow f(x) \text{ is ultralarge}$$

*or*

$$\lim_{x \to a_+} f(x) = \pm\infty$$

*If it is the case for $x$ less than $a$, we write*

$$x \simeq a_- \Rightarrow f(x) \text{ is ultralarge}$$

*or*

$$\lim_{x \to a_-} f(x) = \pm\infty$$

**Example:** The function $f : x \mapsto 1/x$ has a vertical asymptote at 0. The only parameter of the function is 1, always observable. If $dx$ is a positive ultrasmall number then $f(dx)$ is positive ultralarge. Hence

$$\frac{1}{dx} \text{ is ultralarge}$$

We also extend properties of limits to cases where $x$ is positive ultralarge or negative ultralarge, written $x \to +\infty$ or $x \to -\infty$

**Definition 19**
*A real function $f$ defined on an interval of the form $[b, +\infty[$ or $]-\infty, b]$ has a **horizontal asymptote at** $+\infty$ (resp. $-\infty$) if there is an observable number $L$ such that*

$$x \to \infty \Rightarrow f(x) \simeq L.$$

*(the same holds for $-\infty$)*

A context is $f$ and $b$, but it is always possible to consider an observable $b$ relative to $f$ hence a context is given by $f$, and $x$ is ultralarge relative to that context. When this situation occurs, we say that $L$ is the limit of $f$ at plus infinity (resp. minus infinity), or that the limit of $f$ is $L$ when $x$ tends to infinity.

We write that $f$ has **a horizontal asymptote** $y = L$ at plus infinity if

$$\lim_{x \to +\infty} f(x) = L.$$

(Similarly for negative infinity.)

**Example:** Consider the limit

$$\lim_{x \to +\infty} \frac{x^2 - 3x + 1}{x^2 + 1}.$$

This means: consider the fraction for an ultralarge value of $x$.

The function $f : x \mapsto \dfrac{x^2 - 3x + 1}{x^2 + 1}$ is defined on $\mathbb{R}$. $1, 2$ and $3$ are always observable. Let $x$ be ultralarge. Then

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 + 1} = \frac{x^2(2 - \frac{3}{x} + \frac{1}{x^2})}{x^2(1 + \frac{1}{x^2})} = \frac{2 - \overbrace{\frac{3}{x}}^{\simeq 0} + \overbrace{\frac{1}{x^2}}^{\simeq 0}}{1 + \underbrace{\frac{1}{x^2}}_{\simeq 0}} \simeq \frac{2}{1} = 2,$$

hence $f$ has a horizontal asymptote $y = 2$ at $\pm\infty$.

We now define the oblique asymptote

**Definition 20**
*A real function $f$ has an **oblique asymptote at** $+\infty$ (resp. $-\infty$) if there exist observable numbers $a, b$ (context is $f$) such that*

$$x \to +\infty \Rightarrow [f(x) - (ax + b)] \simeq 0 \quad (\textit{resp. } x \to -\infty \Rightarrow [f(x) - (ax + b)] \simeq 0).$$

*The line $y = ax + b$ is the **oblique asymptote of** $f$ (at $\pm\infty$).*

The existence of an oblique asymptote is a property of $f$ hence the context is $f$.

This is equivalent to saying that $f(x) \simeq ax + b$ whenever $x$ is ultralarge.

**Example:** Consider

$$f : x \mapsto \frac{x^3 + 2x^2 + x - 1}{x^2 + 1}$$

defined on $\mathbb{R}$. Using long division we have

$$f(x) = x + 2 - \frac{3}{x^2 + 1}.$$

Let $x$ be ultralarge. We have

$$f(x) - (x + 2) = \frac{-3}{x^2 + 1} \simeq 0,$$

because $x^2 + 1$ is ultralarge. Hence $f$ has an oblique asymptote at $y = x + 2$ (at $\pm\infty$), i.e., $a = 1$ and $b = 2$.

**Exercise 99**
Find the asymptotes (if any) of

(1) $f : x \mapsto \dfrac{x}{2x^2 + 1}$

(2) $g : x \mapsto \dfrac{2x^2 + 1}{x}$

(3) $h : x \mapsto \dfrac{x^3 + 2}{2x^2 - 1}$

(4) $i : x \mapsto \dfrac{x^2 + 2x + 1}{x + 1}$

(5) $j : x \mapsto \dfrac{3x^3 + 2x^2 - x + 12}{x^2 + 8}$

---

For functions which are not rational functions, where the polynomial long division does not apply, we have the following:

**Theorem 30**
*Let $f$ be a real function and let $a$ and $b$ be observable (context is $f$). Then $f$ has an oblique asymptote at $y = ax + b$ at $+\infty$ (resp. $-\infty$) if and only if*

$$\lim_{x\to+\infty} \frac{f(x)}{x} = a \quad \textit{and} \quad \lim_{x\to+\infty} [f(x) - ax] = b.$$

$$\textit{(resp. } \lim_{x\to-\infty} \frac{f(x)}{x} = a \quad \textit{and} \quad \lim_{x\to-\infty} [f(x) - ax] = b.)$$

**Remark:** If $a = 0$ the line $y = ax + b$ becomes $y = b$ i.e., a horizontal asymptote.

**Exercise 100**
Use the definition of limit to rewrite the previous theorem without any reference to limits.

---

**Exercise 101**
Prove the previous theorem.

---

**Example:** Consider $f : x \mapsto \sqrt{x^2 + 1}$ defined on $\mathbb{R}$. Let $x$ be positive ultralarge. Then

$$\frac{f(x)}{x} = \frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2(1+1/x^2)}}{x} = \frac{|x| \overbrace{\sqrt{1+1/x^2}}^{\simeq 1}}{x} \simeq \begin{cases} 1 & \text{it } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Moreover:

$$f(x) - x = \sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1} - x) \cdot (\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \frac{1}{\sqrt{x^2+1} + x} \simeq 0.$$

Hence $f$ has an oblique asymptote at $y = x$ at $+\infty$.
At $-\infty$ the function has an oblique asymptote at $y = -x$.

**Exercise 102**

Find the asymptotes at infinity (if any) of

(1) $f : x \mapsto \dfrac{\sin(x)}{x}$

(2) $g : x \mapsto \dfrac{x^2 + \sin(x)}{x}$

(3) $h : x \mapsto \dfrac{x^2 + \sin(x)}{\sqrt{x}}$

(4) $i : x \mapsto x^{\frac{3}{2}}$

---

**Exercise 103**

Consider a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

where $p$ and $q$ are polynomials. Reminder: the order (or degree) of a polynomial function is the value of the highest exponent of the variable.

(1) In which cases will there be a vertical asymptote?

(2) In which cases will be there be a horizontal asymptote?

(3) In which cases will there be a horizontal asymptote at $y = 0$?

(4) In which cases will there be an oblique asymptote?

---

**Practice exercise 16** Answer page 60

Find all asymptotes of the following functions.

(1) $f_1 : x \mapsto \dfrac{x^2 - x}{x - 1}$

(2) $f_2 : x \mapsto \dfrac{4x^3 + 2x^2 - 5}{3x^3 - 4x^2}$

(3) $f_3 : x \mapsto \sqrt{x^2 + x}$

(4) $f_4 : x \mapsto \dfrac{\sqrt{x^5 + x}}{\sqrt{3x^5 - x}}$

(5) $f_5 : x \mapsto \dfrac{x^2 + 2x}{\sin(x)}$

(6) $f_6 : x \mapsto \dfrac{\sin(x)}{x^2 - x}$

(7) $f_7 : x \mapsto \dfrac{10^x}{10^x + 1}$

# 7 Curve Sketching

Curve sketching needs the following steps:

- Find the domain.
- Find the roots and the intercept (if any).
- Find the asymptotes (if any).
- Find the derivative (if any).
- Find the roots of the derivative (if any).
- Find the second derivative (if any).
- Find the roots of the second derivative (if any).
- Determine the maximums and minimums and bending direction.
- Put all these values in a table.
- Draw arrows which indicate the general direction of the function:
- Use this information to choose a convenient scale.
- Sketch the function.

Reminder: for sketching purposes, the following approximations are good enough: $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$

(1) $f_1 : x \mapsto x^3 + 5x^2 - 8x - 12$ (Check that $-1$ is a root to find the other roots.)

(2) $f_2 : x \mapsto (x - 1) \cdot (x + 1) \cdot x^2$

**Practice exercise 17** Answer page 60

Sketch the following:

(1) $f_1 : x \mapsto \dfrac{x^2}{x + 2}$

(2) $f_2 : x \mapsto x - 1 + \dfrac{9}{x + 1}$

(3) $f_3 : x \mapsto \dfrac{-x^2 - 2x - 1}{x + 3}$

(4) $f_4 : x \mapsto x + 3 + \dfrac{1}{2x + 1}$

(5) $f_5 : x \mapsto \dfrac{x^2 - 4x + 6}{(x-2)^2}$

(6) $f_6 : x \mapsto \dfrac{2x^2 - 3}{x^2 - 1}$

(7) $f_7 : x \mapsto \dfrac{x^2 + 3x - 4}{x^2 - x - 2}$

(8) $f_8 : x \mapsto \dfrac{x^3 + 2}{2x}$

(9) $f_9 : x \mapsto \dfrac{x^3 - 1}{x^2}$

(10) $f_{10} : x \mapsto \dfrac{2x - 1}{\sqrt{x^2 + 2}}$

(11) $f_{11} : x \mapsto \dfrac{\sqrt{x^2 + 1}}{x + 1}$

(12) $f_{12} : x \mapsto \dfrac{\sqrt{x^2 - 4x + 3}}{x + 1}$

## Answers to practice exercises

**Answers to practice exercice 16**, page 57

Vertical asymptote of the form $x = c$, horizontal asymptote of the form $y = b$, oblique asymptote of the form $y = ax + b$.

(1) $y = x$

(2) $y = 1$, $x = 0$, $x = 4/3$

(3) $\begin{cases} y = x & \text{if } x > 0 \\ y = -x & \text{if } x < 0 \end{cases}$

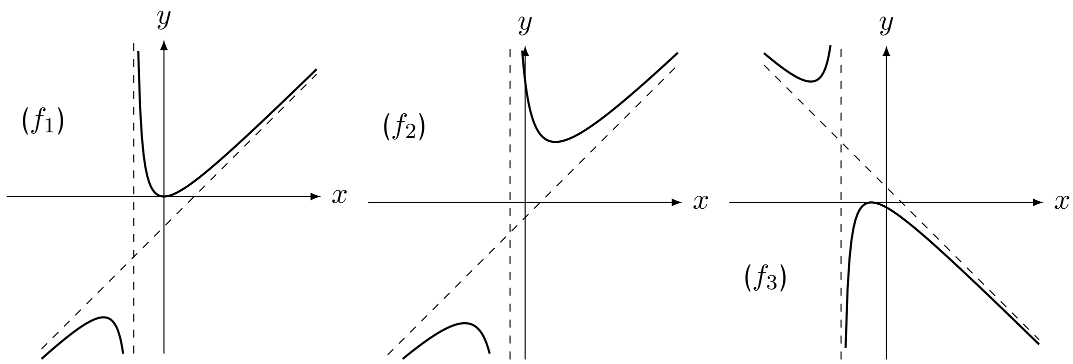
(4) $y = \sqrt{1/3}$, $x = \sqrt[4]{1/3}$

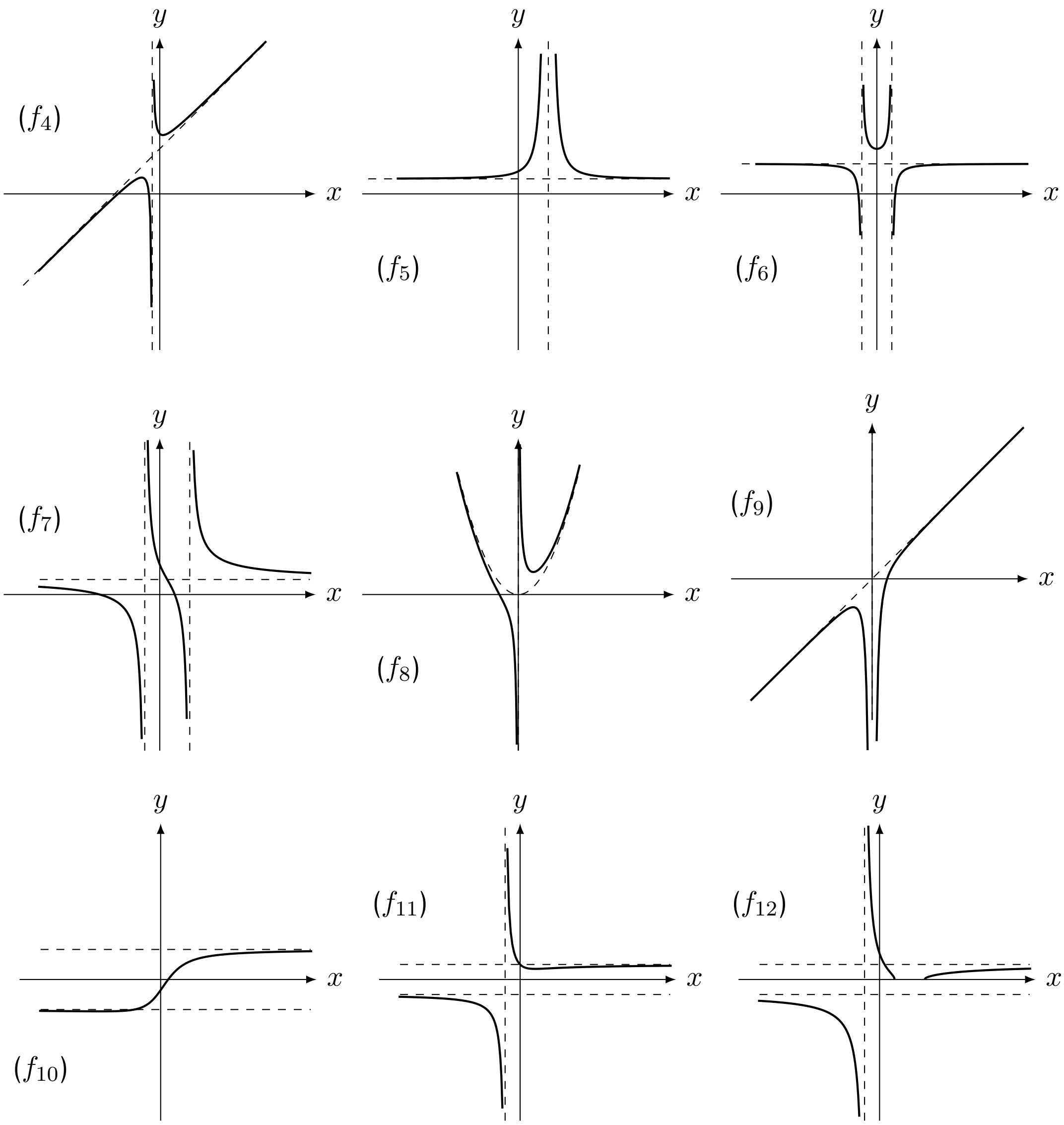
(5) $x = k \cdot \pi \quad k \in \mathbb{Z}$

(6) $y = 0$, $x = 2$

(7) $\begin{cases} y = 0 & \text{if } x < 0 \\ y = 1 & \text{if } x > 0 \end{cases}$

**Answers to practice exercice 17**, page 59



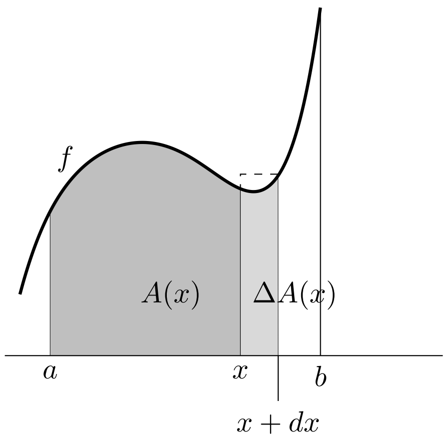

($f_4$)
($f_5$)
($f_6$)
($f_7$)
($f_8$)
($f_9$)
($f_{10}$)
($f_{11}$)
($f_{12}$)
$x$
$y$

# 8
# Integrals

## Area under a curve

Consider a nonnegative function $f$ continuous on a closed interval $[a; b]$. Note $A(x)$ the area between the curve of $f$ and the horizontal $x$-axis.

The variation between $x$ and $x + dx$ is $\Delta A(x)$.




**Exercise 104**

Using the drawing above, consider $f : x \mapsto 3x^2 + x$ between 2 and $2 + dx$.

(1) Write the formula for the variation of the area $\Delta A(2)$ or at least for upper and lower bounds to $\Delta A(2)$.

(2) Determine the equation of $A$.

---

**Theorem 31**

*Let $f$ be a non-negative function continuous on $[a; b]$. Then the function*

$$A : x \mapsto A(x),$$

*where $A(x)$ is the area under the curve of $f$ between $a$ and $x$, has the following properties*

*(1) $A'(x) = f(x)$, whenever $x \in [a; b]$.*

*(2) $A(a) = 0$.*

**Exercise 105**

Prove theorem 31.

Reread exercises 30 and 104 and generalise the proof. At one point you will need the extreme value theorem (theorem 13).

---

**Exercise 106**

Calculate the area under $f : x \mapsto 5x^3 - 2x^2 + x - 2$ between $x = 1$ and $x = 4$.

Use $A' = f$ and $A(1) = 0$.

---

**Exercise 107**

Consider the area under $f$ between $a$ and $b$. Show that if $A' = f$ and $A(a) = 0$, then $A(x) + C$ leads to $C = -A(a)$.

Hence the area is calculated by $A(b) - A(a)$.

---

**Notation**

$$A(b) - A(a) \text{ is written } A(x)\Big|_a^b$$

## A $\int$um of $\int$lices

**Exercise 108**

Total variation of a function:

Let $g : x \mapsto x^2$, $a = 0$ and $b = 5$.

(1) Cut the interval $[a; b]$ into an ultralarge number $N$ of pieces. Put all these pieces together again – add all their lengths. What is the result?

Write this using the symbol for a sum i.e., sum for $k = 0$ to $N - 1$.

(2) For each $dx = \frac{b-a}{N}$ there is a corresponding $\Delta y$. Add all the $\Delta y$ between $f(a)$ and $f(b)$. Find the result.

(3) Use the microscope equation to express $\Delta y$ in terms of $y$ or $y'$. Add all these terms. Find the result.

The (vertical) variation of $f$ between $a$ and $b$ is written $f(x)\Big|_a^b$

---

For the area under $x^2$ between $x = 0$ and $x = 5$, we look at a sum of slices of area. This will give the total variation of the area.

$$A = \sum_{k=0}^{N-1} \Delta A(x_k)$$

This equation is the same as (*) above. Assuming $A' = f$ as shown in theorem 31, we have

$$A(x)\Big|_b^a \simeq \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

Questions: How can we be sure that the function $A$ exists and how do we define the area under a function?

We will now in fact reverse the process: define these sums and then define the area using these.

## Fundamental Theorem of Calculus

**Definition 21**

*Let $f$ be a real function defined on $[a; b]$. Let $n$ be a positive integer. Let $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \ldots, n$. We say that $f$* ***is integrable on*** *$[a; b]$ if there is an observable $I$ such that for any ultralarge integer $n$ with $dx = \dfrac{b-a}{n}$ and $x_i = a + i \cdot dx$, for $i = 0, \ldots, n$, we have*

$$\sum_{i=0}^{n-1} f(x_i) \cdot dx \simeq I.$$

*If such an $I$ exists, it is called the integral of $f$ between $a$ and $b$; written*

$$\int_a^b f(x) \cdot dx.$$

Note that this sum is defined whether $f$ is positive or not.

## preliminary results

**Exercise 109**

Prove the following preliminary results

---

**Lemma 1**

*Let $dx = \frac{b-a}{N}$ for ultralarge $N$, and all $\varepsilon_i \simeq 0$. Then*

$$\sum_{i=0}^{N-1} \varepsilon_i \cdot dx \simeq 0$$

**Lemma 2**

*Let $f$ be an function continuous on $[a; b]$. Let $\frac{1}{N} \simeq 0$, $dx = \frac{b-a}{N}$ and $x_k = a + k \cdot dx$, then there exists a point $c \in [a; b]$ such that*

$$f(c) \cdot (b-a) = \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

**Lemma 3**
*If $f$ is continuous on $[a,b]$ and $u$ and $v$ in $[a,b]$, then $u \simeq v \Rightarrow f(u) \simeq f(v)$*

**Theorem 32**
*If $f$ is continuous an $[a;b]$ then $f$ is integrable on $[a;b]$*

**Exercise 110**
Difficult!
To prove theorem 32, you must show that

(1) the observable neighbour of the sum exists, and

(2) this observable neighbour does not depend on the choice of $N$.

that for $\frac{1}{N} \simeq 0$ and $\frac{1}{M} \simeq 0$ with $du = \frac{b-a}{N}$ and $u_k = a + k \cdot du$ and also $dv = \frac{b-a}{M}$ and $v_j = a + j \cdot dv$ then

$$\sum_{k=0}^{N-1} f(u_k) \cdot du \simeq \sum_{j=0}^{N-1} f(v_j) \cdot dv$$

This can be done by using $\sum_{i=0}^{N \cdot M-1} f(w_i) \cdot dw$ with $dw = \frac{b-a}{M \cdot M}$ and comparing each sum with this one.

By symmetry, it is enough to show that

$$\sum_{k=0}^{N-1} f(u_k) \cdot du \simeq \sum_{i=0}^{N \cdot M-1} f(w_i) \cdot dw$$

Consider an interval $[u_\ell; u_{\ell+1}]$ and *the same interval* $[w_{M \cdot \ell}; w_{M \cdot \ell + M}]$, this interval of length $du$ is one step in the sum of $f(u_k) \cdot dx$ and $M$ steps in the sum of $f(w_i) \cdot dw$.

... and conclude the proof.

---

**Theorem 33 (Continuity of the Integral)**
*If $f$ is continuous on $[a,b]$ then $F(x) = \int_a^x f(t) \cdot dt$ is continuous on $[a,b]$.*

We need to show that $\int_a^x f(t) \cdot dt \simeq \int_a^{x+dx} f(t) \cdot dt$ where $dx \simeq 0$ relative to the context of $f, a$ and $x$.

But for the integral $\int_a^{x+dx} f(t) \cdot dt$ the context is $f, a, x$ and also $dx$, hence we need to use an extra context of ultrasmallness! We write $\frac{1}{N} \overset{+}{\simeq} 0$ to indicate an ultralarge relative to this extended context. We can use the same $N$ for the first integral since integrability means that it does not matter which $N$ is chosen provided it is ultralarge (theorem 32).

The idea is to divide the intervals $[a,x]$ and $[a,x+dx]$ into the same number of pieces.

Since $x$ is a constant here, we will use $t \in [a,x]$ and $u \in [a,x+dx]$ as variables.

**Exercise 111**

Try to complete the proof

---

**Theorem 34 (Additivity of the integral)**
*Let $f$ be a real integrable function continuous on $[a;c]$ and $b \in [a;c]$. Then*

$$\int_a^b f(x) \cdot dx + \int_b^c f(x) \cdot dx = \int_a^c f(x) \cdot dx.$$

**Exercise 112**

Prove theorem 34.

---

**Theorem 35**
*If $f$ is a continuous function on $[a,b]$ then*

$$F(x) = \int_a^x f(t) \cdot dt$$

*is an antiderivative of $f$ on $]a,b[$ and the only one satisfying $F(a) = 0$.*

**Exercise 113**

Prove theorem 35 starting with the definition of the derivative applied to the integral. By theorem 32, it is integrable.

---

**Theorem 36 (Fundamental theorem of Calculus)**
*Let $f$ be a function continuous on $[a;b]$. Let $F$ be an antiderivative of $f$ on $[a;b]$. Then*

$$\int_a^b f(x) \cdot dx = F(b) - F(a).$$

The method used in the proof can also be seen as looking at the link between the global variation of a function $F$ and its derivative $f$.

**Exercise 114**

Consider the variation of $F$ between $a$ and $b$.
Let $n \in \mathbb{N}$ such that $1/N \simeq 0$ and $dx = \frac{b-a}{N}$ and $x_k = a + k \cdot dx$.
Then clearly, we have

$$F(b) - F(a) = \sum_{k=0}^{N-1} \Delta F(x_k)$$

Here the context is $f, a, b$ – not necessarily any given $x_j$!

(1) On each interval $[x_k, x_{k+1}]$ (which is also in the form $[x_k, x_k + dx]$) there is a $c$ such that

$$F(x_k + dx) - F(x_k) = f(c) \cdot dx,$$

Why is this? By what theorem?

(2) Explain why we have $f(c) \simeq f(x_k)$.

(3) Conclude by explaining why:

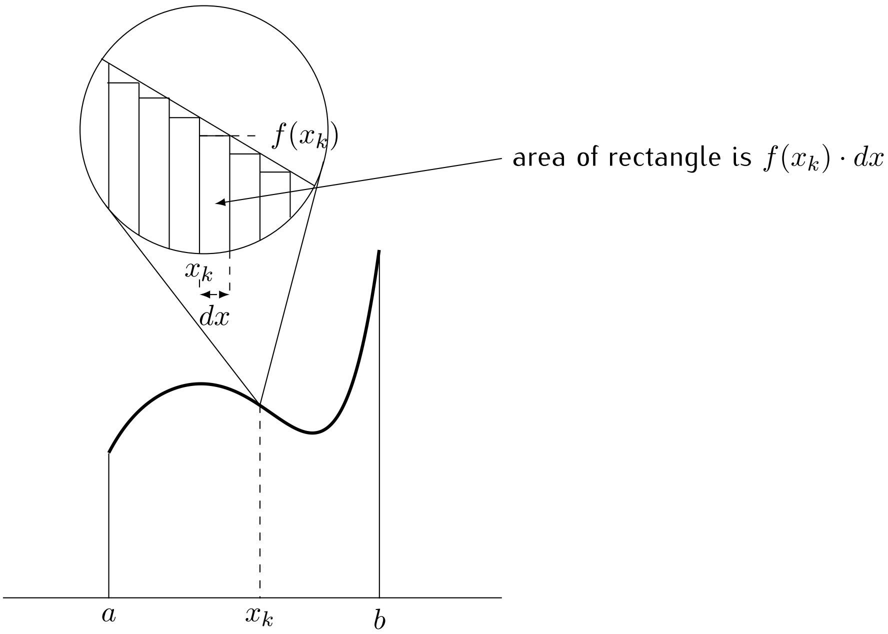
$$\sum_{k=0}^{N-1} F(x_k + dx) - F(x_k) = \sum_{k=0}^{N-1} f(x_k) \cdot dx + \sum_{k=0}^{N-1} \varepsilon_k \cdot dx \simeq \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

---

Hence, the global variation of $F$ between $a$ and $b$ is, up to an ultrasmall value, the sum of $F'(x_i) \cdot dx$ provided $F'$ is continuous on $[a, b]$.

Page 63 we looked at one slice of the area under a positive function. Now we show that if we sum up all slices on the area under a curve, the antiderivative gives the answer. Hence we have

$$\text{area} \simeq \sum_{i=0}^{N-1} f(x_k) \cdot dx.$$




⚠ The drawing can be misleading. It is only a specific case. A continuous function does not necessarily appear as a straight line under magnification. The extreme value theorem ensures that it has a maximum and minimum on the interval.

**Notation**: we write

$$F(x)\Big|_a^b = F(b) - F(a).$$

If bounds are given, the integral represents a value: it is a **definite integral**. If no bounds are given, it represents an antiderivative: it is an **indefinite integral**.

**Exercise 115**

Show that for a definite integral, it does not matter which antiderivative is chosen.

---

**Exercise 116**

What conditions would a function need to satisfy in order to be non-integrable? Give such a function.

---

**Exercise 117**

A constant function $f : x \mapsto C$ from $a$ to $b$ defines a rectangle. Check that the area under $f$ is the "usual" formula: $(b - a) \cdot C$

---

**Exercise 118**

The function $y = x$ defines a triangle. Show that the area of the triangle from 0 to $a$ yields the "usual" result for the area of a triangle.

---

**Exercise 119**

(1) Calculate the area between the curve and the $x$-axis for $y = x^2$ from $x = -5$ to $x = 5$.

(2) Calculate the area between the curve and the $x$-axis for $y = x^3$ from $x = 0$ to $x = 3$.

(3) Calculate the area between the curve and the $x$-axis for $y = x^3$ from $x = -2$ to $x = 0$.

(4) Calculate the area between the curve and the $x$-axis for $y = x^3$ from $x = -10$ to $x = 10$.

---

Notice that the integral can be a negative value. If $f$ represents the velocity of an object, a negative integral means that the distance is becomming smaller. If the integral is equal to zero, the object is back where it started.

So far we have assumed that an area function exists. Now we can give a definition.

**Definition 22 (Area)**
*The area between a positive continuous function and the $x$-axis, on an interval $[a; b]$ is given by the integral of the function on $[a; b]$.*

**Exercise 120**

Calculate the mean value of $x \mapsto x^2$ on $[-4; 4]$.

---

## Linearity

**Theorem 37 (Linearity of the integral)**
*Let $f$ and $g$ be real functions continuous on $[a;b]$. Let $\lambda, \mu$ be real numbers. Then*

*(1)*
$$\int_a^b (\lambda \cdot f(x)) \cdot dx = \lambda \cdot \int_a^b f(x) \cdot dx$$

*(2)*
$$\int_a^b (f(x) + g(x)) \cdot dx = \int_a^b f(x) \cdot dx + \int_a^b g(x) \cdot dx.$$

Note that if $f$ and $g$ are integrable then all linear combinations of $f$ and $g$ are integrable.

**Theorem 38 (Monotonicity of the integral)**
*Let $f$ be a real function continuous on $[a;b]$.*

*(1) If $f(x) \geq 0$ (resp. $> 0$) for each $x \in [a;b]$ then*
$$\int_a^b f(x) \cdot dx \geq 0 \quad (\textit{resp.} > 0).$$

*(2) If $f(x) = 0$ for each $x \in [a;b]$ then*
$$\int_a^b f(x) \cdot dx = 0.$$

*(3) If $f(x) \leq 0$ (resp. $< 0$) for each $x \in [a;b]$ then*
$$\int_a^b f(x) \cdot dx \leq 0 \quad (\textit{resp.} < 0).$$

**Exercise 121**
Prove theorems 37 and 38.

---

**Exercise 122**
Prove theorem 39.

---

**Theorem 39 (Integration by parts)**
*Let $f$ and $g$ be real functions continuous on $[a;b]$ such that $f'$ and $g'$ are continuous on $[a;b]$. Then*
$$\int_a^b f'(x) \cdot g(x) \cdot dx = f(x) \cdot g(x)\Big|_a^b - \int_a^b f(x) \cdot g'(x) \cdot dx.$$

**Example:** Consider the integral
$$\int_0^{\pi/2} x \cdot \sin(x) \cdot dx.$$
To integrate by parts, use $f' : x \mapsto \sin(x)$ et $g : x \mapsto x$. We have $f(x) = -\cos(x)$ and $g'(x) = 1$, hence
$$\int_0^{\pi/2} x \cdot \sin(x) \cdot dx = -x \cdot \cos(x)\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos(x) \cdot dx = \sin(x)\Big|_0^{\pi/2} = 1.$$
We also deduce that
$$\int x \cdot \sin(x) \cdot dx = -x \cdot \cos(x) + \sin(x) + C.$$

**Exercise 123**

Use integration by parts to compute the following integrals:

(1) $\int x \cdot \cos(x) \cdot dx$

(2) $\int (\cos(x))^2 \cdot dx$

(3) $\int x^2 \cdot \sin(x) \cdot dx$

(4) $\int \sin(x) \cdot \cos(x) \cdot dx$

---

**Exercise 124**

For each of the following functions, find an antiderivative:

(1) $f : t \mapsto 3t^2 + 1$

(2) $f : t \mapsto 4 - 3t^3$

(3) $f : s \mapsto 7s^{-3}$

(4) $f : x \mapsto (x - 6)^2$

(5) $f : y \mapsto y^{\frac{3}{2}}$

(6) $f : x \mapsto |x|$

(7) $f : u \mapsto u^2 + u^{-2}$

(8) $f : x \mapsto 4$

(9) $f : t \mapsto t$

(10) $f : z \mapsto \dfrac{2}{z^2}$

Check your results by differentiating them.

---

**Exercise 125**

(1) If $F'(x) = x + x^2$ for all $x$, find $F(1) - F(-1)$.

(2) If $F'(x) = x^4$ for all $x$, find $F(2) - F(1)$.

(3) If $F'(t) = t^{\frac{1}{3}}$ for all $t$, find $F(8) - F(10)$.

---

**Exercise 126**

The following computation may seem correct: $\int_{-1}^{1} x^{-2}dx = -\frac{1}{x}\Big|_{-1}^{1} = -2$ yet there is no $x \in [-1,1]$ such that $f(x) < 0$. By theorem 38 we should therefore have a positive value for the integral. Why is this not so?

---

**Theorem 40 (Integration with inside derivative)**
*Let $f$ and $g$ be real functions differentiable on $[a;b]$ such that $f'$ and $g'$ are continuous on $[a;b]$. Then*

$$\int_a^b f'(g(x)) \cdot g'(x) \cdot dx = f(g(x))\Big|_a^b.$$

**Exercise 127**

Prove theorem 40.

---

**Exercise 128**

Compute the following integrals:

(1) $\displaystyle\int 2x \cdot \sin(x^2) \cdot dx$

(2) $\displaystyle\int x^2 \cdot (x^3+1) \cdot dx$

(3) $\displaystyle\int \sin(x) \cdot \cos(\cos(x)) \cdot dx$

(4) $\displaystyle\int \sin(x) \cdot \cos^2(x) \cdot dx$

---

## Variable substitution

Consider $\displaystyle\int_a^b f(x) \cdot dx$.

If $x$ is a function of $u$ written $x = g(u)$ then $dx = g'(u) \cdot du$,

$f(x)$ becomes $f(g(u))$ and the limits must be changed to $a_1$ and $b_1$ so that $g(a_1) = a$ and $g(b_1) = b$

**Example:** Let

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx.$$

Consider the variable change $u = 1 + \sqrt{x}$. Then $x = (u-1)^2 = g(u)$, the derivative of $g$ is continuous. If $x = 0$ then $u = 1$ and if $x = 1$ then $u = 2$. Moreover $f(g(u)) = \sqrt{u}$ and

$$dx = 2 \cdot (u-1) \cdot du.$$

Replacing all terms we obtain

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx = 2\int_1^2 \sqrt{u} \cdot (u-1) \cdot du = 2\int_1^2 \left(u^{3/2} - u^{1/2}\right) \cdot du$$

so that

$$2\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_1^2=\frac{8+8\sqrt{2}}{15}.$$

As $g$ has an inverse which is $x\mapsto 1+\sqrt{x}$ and is differentiable (except at $x=0$), we can revert to the variable $x$ and find an antiderivative:

$$\int\sqrt{1+\sqrt{x}}\cdot dx=\frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5-\frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3+C.$$

**Exercise 129**

Calculate

$$\int_0^1\sqrt{5x+2}\cdot dx.$$

Use $u=5x+2$. Calculate $du$, change the bounds, calculate the integral.

Same integral. Use $v=\sqrt{5x+2}$

---

The difficulty is usually to find which variable substitution is best.

**Exercise 130**

Use variable substitution to evaluate the following:

(1) $\displaystyle\int_0^{10}\frac{1}{(2x+2)^2}\cdot dx$

(2) $\displaystyle\int(3-4z)^6\cdot dz$

(3) $\displaystyle\int_{-1}^1 2t\sqrt{1-t^2}\cdot dt$

(4) $\displaystyle\int_a^b\sqrt{3y+1}\cdot dy$

(5) $\displaystyle\int\frac{4y}{(2+3y^2)^2}\cdot dy$

(6) $\displaystyle\int_{-2}^2 x(4-5x^2)^2\cdot dx$

(7) $\displaystyle\int(1-x)^{\frac{3}{2}}\cdot dx$

---

**Practice exercise 18** Answer page 77

(1) $\displaystyle\int_0^1\frac{u}{\sqrt{1-u^2}}\cdot du$

(2) $\displaystyle\int_1^2\frac{u}{\sqrt{1-u^2}}\cdot du$

(3) $\displaystyle\int_0^1\sqrt{1+\sqrt{x}}\cdot dx$

(4) $\displaystyle\int_0^{10}t(t^2+3)^{-2}\cdot dt$

(5) $\displaystyle\int_{\sqrt{6}}^5 x(x^2+2)^{\frac{1}{3}}\cdot dx$

(6) $\displaystyle\int_{-1}^1\frac{x^2}{(4-x^3)^2}\cdot dx$

(7) $\displaystyle\int_1^2\frac{1}{t^2\sqrt{1+\frac{1}{t}}}\cdot dt$

Variable substitution is formalised in the following theorem.

**Theorem 41 (Integration by variable substitution)**
*Let $f$ be a real function continuous on $[a;b]$. Let $g$ be a function whose derivative is continuous and such that for $e, d \in \mathbb{R}$ we have $g(d) = a$ and $g(e) = b$. Then*

$$\int_a^b f(x) \cdot dx = \int_d^e f(g(u)) \cdot g'(u) \cdot du.$$

This formula looks probably quite difficult, but hopefully, the exercises done above show that it amounts to a systematic procedure.

A simplified writing can be used: we have already used the writing $y = f(x)$ where $y$ is a dependent variable and $x$ the independent variable. When several functions are used, we can write $u = f(x)$ and $v = g(x)$, then we have (for constant $c$ and for $U' = u$ and $V' = v$):

- $c' = 0$
- $(c \cdot u)' = c \cdot u'$
- $(u + v)' = u' + v'$
- $(u \cdot v)' = u' \cdot v + u \cdot v'$
- $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$
- $(u \circ v)' = u' \cdot v'$ (in this case, $u$ depends on $v$ which depends on $x$).
- $(x^n)' = nx^{n-1}$
- $\sin'(x) = \cos(x)$
- $\cos'(x) = -\sin(x)$
- $\tan'(x) = 1 + \tan^2(x) = \frac{1}{\cos^2(x)}$
- $\int c \cdot u \cdot dx = c \cdot U + k$
- $\int (u + v) \cdot dx = U + V + k$
- $\int u(v) \cdot v' \cdot dx = U(v) + k$
- $\int u' \cdot v \cdot dx = u \cdot v - \int u \cdot v' \cdot dx$

## Answers to practice exercises

**Answers to practice exercice 11**, page 47

(1) $f'(x) = 20x^3 + 3x^2 - 4x$

(2) $g'(x) = 10\sqrt{3}x$

(3) $h'(x) = -\dfrac{x^4 + 4x^3 - 3x^2 + 10x + 10}{(x^3 - 5)^2}$

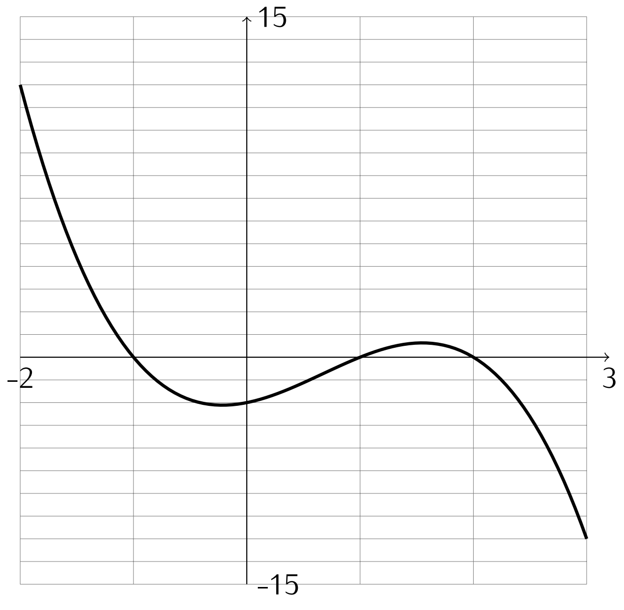
(4) $j'(x) = 20x^3 - \dfrac{6x - 2}{(3x^2 - 2x + \pi)^2}$

(5) $k'(x) = 0$

(6) $l'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} - \dfrac{3}{x^4} - \dfrac{4}{x^5}$

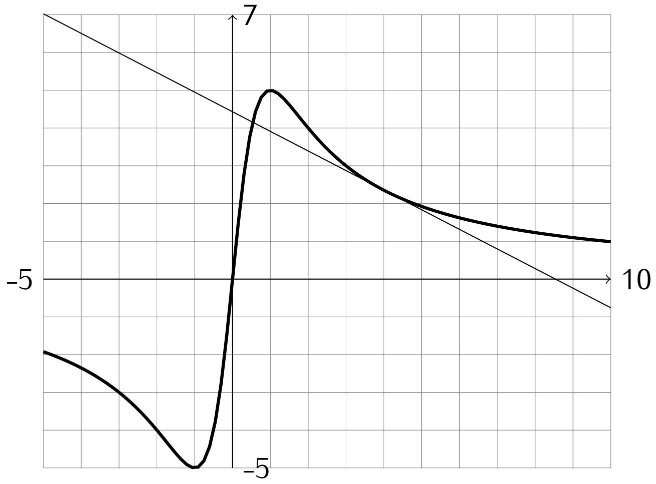
(7) $m'(x) = \dfrac{(x^2 + x + 1)(3x^2 + 2x) - (x^3 + x^2)(2x + 1)}{(x^2 + x + 1)^2} = \dfrac{x(x^3 + 2x^2 + 4x + 2)}{(x^2 + x + 1)^2}$

**Answers to practice exercice 12**, page 47




**Answers to practice exercice 13**, page 47

Tangent line is $y = -\dfrac{4}{5}x + \dfrac{27}{5}$




**Answers to practice exercice 14**, page 47

(1) $3x^2 + 2x + 2$

(2) $-3x^2 + 4x - 2$

(3) $x^2 - 5x + 6$

(4) $(x - 2)^2$

(5) $\dfrac{x(x+4)}{(x+2)^2}$

(6) $\dfrac{x^2 + 2x - 8}{(x+1)^2}$

(7) $\dfrac{4x^2 + 4x - 3}{(2x+1)^2}$

(8) $-\dfrac{x^2 + 6x + 5}{(x+3)^2}$

(9) $\begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \\ \text{not differentiable} & \text{if } x = 2 \end{cases}$

(10) $\begin{cases} \dfrac{x(x+4)}{(x+2)^2} & \text{if } x \geq 0 \\ \dfrac{-x(x-4)}{x-2)^2} & \text{if } x \leq 0 \end{cases}$

(11) $\dfrac{x^2 + 2x + 2}{(x+1)^2}$

(12) $\begin{cases} 3x^2 - 12x + 11 & \text{if } x \in ]1;2[\cup]3;\infty[ \\ -3x^2 + 12x - 11 & \text{if } x \in ]-\infty;1[\cup]2;3[ \\ \text{not differentiable} & \text{if } x \in \{1;2;3\} \end{cases}$

**Answers to practice exercice 15**, page 48

(1) $f_1' : x \mapsto \dfrac{9x^2 + 2}{2\sqrt{3x^3 + 2x + 1}}$

(2) $f_2' : x \mapsto 10x \cdot (x^2 + 3)^4$

(3) $f_3' : x \mapsto an \cdot (ax + b)^{n-1}$

(4) $f_4' : x \mapsto \dfrac{3x^2}{2\sqrt{x^3 + 1}}$

(5) $f_5' : x \mapsto \cos(x^2 + 3x) \cdot (2x + 3)$

(6) $f_6' : \theta \mapsto -6\cos(3\theta) \cdot \sin(3\theta)$

(7) $f_7' : u \mapsto \cos(\sin(u)) \cdot \cos(u)$

(8) $f_8' : x \mapsto 8x\tan(\tan^2(x^2)(1+\tan^2(\tan^2(x^2))(\tan(x^2)(1+\tan^2(x^2))$

(9) $f_9 : v \mapsto -\sin(v)$

(10) $f_{10}' : x \mapsto 0$

**Answers to practice exercice 18**, page 73

(1) 1 Use $x = 1 - u^2$.

(2) undefined – for $u > 1$ we have the square root of a negative number.

(3) $\frac{8(\sqrt{2}+1)}{15}$ Use $u = 1 + \sqrt{x}$

(4) $\frac{50}{309}$ Use $u = t^2 + 3$

(5) $\frac{195}{8}$ Use $u = x^2 + 2$

(6) $\frac{2}{45}$ Use $u = 4 - x^3$

(7) $-\sqrt{6} + 2\sqrt{2}$ Use $u = 1 + \frac{1}{t}$

# 9
# limits

A function $f$ is defined on the left of $a$ (resp. on the right) if $f(x)$ is defined for all $x \simeq a$ with $x < a$ (resp. $x > a$). It is clear that $f$ is defined around $a$ if and only if $f$ is defined on the right and on the left of $a$.

**Definition 23 (One sided Continuity)**
*Let $f$ be a real function and $a \in \mathbb{R}$.*

(1) *Suppose that $f$ is defined on the left of $a$. Then $f$ is* ***continuous on the left at*** *$a$ if $x < a$ and $x \simeq a \implies f(x) \simeq f(a)$.*

(2) *Suppose that $f$ is defined on the right of $a$. Then $f$ is* ***continuous on the right at*** *$a$ if $x > a$ and $x \simeq a \implies f(x) \simeq f(a)$.*

It is immediate that $f$ is continuous at $a$ if and only if it is continuous on the right and on the left at $a$.

We now extend the concept of continuity at a point to continuity on an interval.

**Exercise 131**

Prove directly that $x \mapsto \sqrt{x}$ is continuous on its domain i.e, for any value $x = a$ in the domain.

Hint: start by the definition, then multiply and divide by $(\sqrt{a+dx} + \sqrt{a}$.

---

If we want to study the behaviour of $f$ in the neighbourhood of $a$, the function $f$ must be defined *around* $a$, but not necessarily at $a$. If the function is defined in a neighbourhood of $a$, by closure, it is possible to use a neighbourhood defined by observable bounds. Hence $f(x)$ must exist for $x \simeq a$ but $f(a)$ does not necessarily exist. Context is $f$ and $a$.

**Definition 24**
*A* ***deleted interval of*** *$a$ is an interval around $a$ not containing $a$.*

| The limit of $f$ at $a$ is the value that $f$ should take in order to be continuous at $a$. |
|---|

**Definition 25**
*Let $f$ be a real function defined on a deleted interval of $a$. Context is $f$ and $a$. We say that $f$* ***has a limit at*** *$a$ if there exists an observable number $L$ such that if we had $f(a) = L$ then $f$ would be continuous at $a$,*

In other terms, if there is an observable number $L$ such that

$$x \simeq a \Longrightarrow f(x) \simeq L.$$

Of course, by this definition, if $f$ *is* continuous at $a$, then the limit of $f$ at $a$ is $f(a)$.

| The limit of $f$ at $a$ is the observable value of $f(x)$ when $x \simeq a$ |
| --- |

The definition of limit can also be interpreted in the following way:

If $f$ has a limit at $a$ then it is the observable neighbour of $f(a + dx)$.
If $L$ is the limit of $f$ at $a$ we write

$$f(a + dx) \simeq L$$

or

$$\lim_{x \to a} f(x) = L,$$

or

$$\lim_{h \to 0} f(a + h) = L.$$

**Exercise 132**

Calculate

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3}.$$

Show that it is equal to

$$\lim_{h \to 0} \frac{2(3 + h)^2 - 7(3 + h) + 3}{(3 + h) - 3}.$$

---

**Exercise 133**

Consider the signum function sgn, defined by

$$\text{sgn} : x \mapsto \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

Check that sgn is defined around 0. Does it have a limit at 0?

---

## One Sided Limits

A function is defined **on the left** (respectively **on the right**) of $a$, if $f(x)$ exists for $x \simeq a$, $x < a$ (respectively $x \simeq a$, $x > a$).

**Definition 26**
*Let $f$ be a real function defined on the left of $a$. The function $f$ **has a limit on the left of** $a$ if there is an obervable number $L$ such that*

$$x \simeq a \text{ and } x < a \implies f(x) \simeq L.$$

If the limit on the left exists it is unique (it is the observable neighbour of $f(x)$). We write:

$$\lim_{x \to a_-} f(x) = L, \quad \text{or} \quad x \simeq a_- \Rightarrow f(x) = L.$$

The symbol $a_-$ indicates that we choose numbers less than $a$.
Similarly we define the **limit on the right of** $a$ and write:

$$\lim_{x \to a_+} f(x) = L, \quad \text{or} \quad x \simeq a_+ \Rightarrow f(x) = L.$$

The symbol $a_+$ indicates that we choose numbers greater than $a$.

**Exercise 134**
Consider $f$ defined by

$$f : x \mapsto \sin(1/x), \quad \text{for } x > 0.$$

Check that $f$ is defined on the right of 0.
Does it have a limit on the right of zero?

---

Using limits, the derivative may be re-defined in the following way:

Let $f$ be a real function defined on an interval containing $a$. **The derivative of** $f$ **at** $a$ is the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists. If it exists, it is noted $f'(a)$. It is the derivative of $f$ at $a$ and $f$ is said to be **differentiable** at $a$.

> The limit is only a rewriting. The "equal" sign used is there to say that the limit is the value that the function can be ultraclose to.
> When a limit appears in a problem, the first thing to do is to rewrite it in terms of ultracloseness.

We extend the definition of limit to the cases where the function reaches ultralarge values.

⚠ Introducing a new symbol: if relative to a context, we consider ultralarge values of $x$ or ultralarge values of $f(x)$, the infinity symbol "$\infty$" is used. But **no value can ever be equal to** $\infty$.

⚠ The $\infty$ symbol cannot be used in operations, because it is not a number.

**Definition 27**
*Let $f$ be a real function defined on a deleted interval of $a$. The context is $f$ and $a$. We say that $f$ **tends to plus infinity** ($+\infty$) (resp. minus infinity ($-\infty$)) at $a$ if $f(x)$ is positive ultralarge (resp. negative ultralarge) whenever $x \simeq a \quad x \neq a$*
*written*

$$\lim_{x\to a} f(x) = \infty$$

*The definition for one-sided limits is similar.*

Similarly

$$\lim_{x\to\infty} f(x) = L$$

stands for: there is an observable $L$ such that $f(x) \simeq L$ whenever $x$ is ultralarge.

**Theorem 42 (Rule of de l'Hospital for $0/0$ )**
*Let $f$ and $g$ be differentiable functions at $a$. Suppose that $f(a) = g(a) = 0$, but that $g'(a) \neq 0$. Then*

$$\frac{f(a+dx)}{g(a+dx)} \simeq \frac{f'(a)}{g'(a)}$$

*(provided $f'(a)$ and $g'(a)$ exist).*

**Exercise 135**
Prove theorem 42.

---

The rule of de l'Hospital also holds for the case where $a$ is ultralarge. And more generally

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if $lim_{x\to a} g'(x) \neq 0$.

**Exercise 136**
Evaluate using de L'Hospital's rule.

$$\frac{x-1}{\sqrt{x^2-1}}$$

for $x \simeq 1$.

---

**Exercise 137**
Assuming the rule of de l'Hospital holds for the case $\frac{\text{ultrasmall}}{\text{ultrasmall}}$, show that it holds for the case $\frac{\text{ultralarge}}{\text{ultralarge}}$

---

**Exercise 138**
Evaluate using de L'Hospital's rule.

(1) $\dfrac{1/t-1}{t^2-2t+1}$ for $t \simeq 1$ (with $(t>1)$).

(2) $\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ for $x \simeq 1$.

(3) $\dfrac{x^2}{\sqrt{2x+1}-1}$ for $x \simeq 0$.

(4) $\dfrac{2+1/t}{3-2/t}$ for $t \simeq 0$.

(5) $\dfrac{x+5-2x^{-1}-x^{-3}}{3x+12-x^{-2}}$ for ultralarge $x$

(6) $\left(t+\dfrac{1}{t}\right)((4-t)^{3/2}-8)$ for $t \simeq 0$.

(7) $\dfrac{u+u^{-1}}{1+\sqrt{1-u}}$ for ultralarge $u$.

---

**Practice exercise 19** Answer page 96

Calculate the following limits. The answer should be a number, $+\infty$, $-\infty$ or "does not exist"

(1) $\lim\limits_{x\to\infty} \dfrac{6x-4}{2x+5}$

(2) $\lim\limits_{x\to\infty} x^3-10x^2-6x-2$

(3) $\lim\limits_{x\to\infty} \dfrac{x^2-x+4}{3x^2+2x-3}$

(4) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+2}}{\sqrt{3x+1}}$

(5) $\lim\limits_{x\to\infty} x-\sqrt{x}$

(6) $\lim\limits_{x\to\infty} \sqrt[3]{x+2}$

(7) $\lim\limits_{x\to 0_-} 1+\dfrac{1}{x}$

(8) $\lim\limits_{x\to 0} \dfrac{1}{x^2}-\dfrac{1}{x}$

(9) $\lim\limits_{x\to 0} \dfrac{1+2x^{-1}}{7+x^{-1}-5x^{-2}}$

(10) $\lim\limits_{x\to 2} \dfrac{1-x}{2-x}$

(11) $\lim\limits_{x\to 3_+} \dfrac{x+1}{(x-2)(x-3)}$

(12) $\lim\limits_{x\to 3} \dfrac{x+1}{(x-2)(x-3)}$

(13) $\lim\limits_{x\to 1} \dfrac{3x^2+4}{x^2+x-2}$

(14) $\lim\limits_{x\to 2_+} \dfrac{x^2+4}{x^2-4}$

(15) $\lim\limits_{x\to\infty} \sqrt{x^2+1}-x$

(16) $\lim\limits_{x\to-\infty} \sqrt{x^2+1}-x$

(17) $\lim\limits_{x\to\infty} \sqrt{x^2-3x+2}-\sqrt{x^2+1}$

(18) $\lim\limits_{x\to\infty} \sqrt[3]{x+4}-\sqrt[3]{x}$

**Practice exercise 20** Answer page 96

Evaluate using de L'Hospital's rule.

(1) $\lim\limits_{x\to 0} \dfrac{\sqrt{9+x}-3}{x}$

(2) $\lim\limits_{x\to 2} \dfrac{2-\sqrt{x+2}}{4-x^2}$

(3) $\lim\limits_{u\to\infty} \dfrac{\sqrt{u+1}+\sqrt{u-1}}{u}$

(4) $\lim\limits_{x\to 0} \dfrac{(1-x)^{1/4}-1}{x}$

(5) $\lim_{t \to 0_+} \left( \frac{1}{t} + \frac{1}{\sqrt{t}} \right) (\sqrt{t+1} - 1)$

(6) $\lim_{u \to 1} \frac{(u-1)^3}{u^{-1} - u^2 + 3u - 3}$

(7) $\lim_{u \to 0_+} \frac{1 + 5/\sqrt{u}}{2 + 1/\sqrt{u}}$

(8) $\lim_{x \to \infty} \frac{x + x^{1/2} + x^{1/3}}{x^{2/3} + x^{1/4}}$

(9) $\lim_{t \to \infty} \frac{1 - t/(t-1)}{1 - \sqrt{t/(t-1)}}$

# 10

# More on integration

**Definition 28**
*The $\infty$ symbol in the bounds of an integral indicates a limit.*

$$\int_a^{\infty} f(x) \cdot dx = \lim_{n \to \infty} \int_a^n f(x) \cdot dx$$

This is calculated by taking ultralarge $N$ in $\int_a^N$ and taking the observable part of the result (if it exists and is independent of $N$).

**Exercise 139**

Check that an derivative of $x \mapsto \dfrac{x}{x+1}$ is $x \mapsto \dfrac{1}{(x+1)^2}$.

Sketch the curve of $f : x \mapsto \dfrac{1}{(x+1)^2}$ for $x > 0$.

Calculate the area under $f$ between 0 and 10.

Calculate the area under $f$ between 0 and $+\infty$

---

**Exercise 140**

Do infinitely long objects have a finite area?

(1) Calculate the area under $f : x \mapsto \frac{1}{x^2}$ between $x = 1$ and $x = \infty$, i.e: show that this area does not depend on which ultralarge is chosen.

(2) Without any calculation, explain why the total length of both sides (the curve above and the straight line below) is infinite.

(3) Does this prove that a finite amount of paint would be enough to cover the area but not enough to paint the border lines?

---

**Definition 29**
*If the function to integrate is not defined at one of the bounds, then*

$$\int_a^b f(x) \cdot dx = \lim_{u \to a_+} \int_u^b f(x) \cdot dx$$

*or*

$$\int_a^b f(x) \cdot dx = \lim_{u \to b_-} \int_a^u f(x) \cdot dx$$

**Exercise 141**

Evaluate the integrals:

(1) $\displaystyle\int_0^1 2x^{-2} \cdot dx$ (3) $\displaystyle\int_{-1}^2 -5(t+1)^{-1/4} \cdot dt$

(2) $\displaystyle\int_{-2}^3 u^{-3} \cdot du$ (4) $\displaystyle\int_0^4 \frac{1}{2\sqrt{x}} \cdot dx$

---

**Exercise 142**

In the following problems an object moves along the $y$ axis. Its velocity varies with respect to the time. Find how far the object moves between the given times $t_0$ and $t_1$.

(1) $v = 2t + 5$ $t_0 = 0 \quad t_1 = 2$ (4) $v = 3t^2$ $t_0 = 1 \quad t_1 = 3$

(2) $v = 4 - t$ $t_0 = 1 \quad t_1 = 4$

(3) $v = 3$ $t_0 = 2 \quad t_1 = 6$ (5) $v = 10t^{-2}$ $t_0 = 1 \quad t_1 = 100$

---

## Antiderivative of $x \mapsto \dfrac{1}{x}$

Let $n$ be a positive integer. From $(x^{n+1})' = (n+1) \cdot x^n$ we can deduce

$$\int x^n \cdot dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1.$$

Hence an antiderivative of $x \mapsto \dfrac{1}{x}$ is not a particular case of this formula.

**Exercise 143**

Let $f$ be an antiderivative of $x \mapsto \frac{1}{x}$ (why is there one?). Then $f$ is strictly increasing (why?) and so it has an inverse, call it $g$. Show that this implies $g'(x) = g(x)$.

---

**Exercise 144**

Let $a, b > 0$. Use the substitution $u = \frac{t}{a}$ to show that (considering $f$ to be the antiderivative of $\frac{1}{x}$.)

$$\int_a^{a \cdot b} \frac{1}{t} \cdot dt = \int_1^b \frac{1}{u} \cdot du.$$

Deduce that $f(a \cdot b) = f(a) + f(b)$.

---

**Exercise 145**

Let $a > 0$ and $b$ a rational number. Show that (considering $f$ to be the antiderivative of $\frac{1}{x}$.)

$$f(a^b) = b \cdot f(a).$$

(To find the substition, consider the transformation of the bounds.)

---

**Exercise 146**

What kind of function has the properties $f(a \cdot b) = f(a) + f(b)$ and $f(a^b) = b \cdot f(a)$?

---

**Theorem 43**

*The antiderivative $f$ of $\frac{1}{x}$ satisfies the following limits:*

$$\lim_{x \to 0^+} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to +\infty} f(x) = +\infty.$$

**Exercise 147**

Prove theorem 43. Hint: for ultralarge $x$ use ultralarge $N$ such that $2^N \leq x$.

---

**Definition 30**

*The **natural logarithm** is the function* $\ln : ]0; +\infty[ \to \mathbb{R}$ *defined by*

$$x \mapsto \int_1^x \frac{1}{t} \cdot dt.$$

**Definition 31**

*We define $e$ to be the unique number such that*

$$\ln(e) = 1.$$

$e$ is an irrational number whose first digits are

$$e = 2.71828\ldots$$

**Definition 32**

*The **exponential** function* $\exp : \mathbb{R} \longrightarrow ]0; +\infty[$ *is defined as the inverse of ln.*

Thus $\ln$ is in fact $\log_e$ and $\ln(e) = 1$.

We have, for rational $x$, that $a^x = \exp(x \ln(a))$, hence $e^x = \exp(x)$. For irrational $x$, we **define** $a^x$ to be $\exp(x \ln(a))$ hence also $e^x = \exp(x)$ for all $x$.

We also have $\ln(a^y) = y \cdot \ln(a)$ for all $y$. Writing $x = a^y$ we get $\ln(x) = \log_a(x) \cdot \ln(a)$ so $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

**Theorem 44**

*(1) Let $b \in \mathbb{R}$. The function $x \mapsto x^b$ is differentiable on its domain and $(x^b)' = b \cdot x^{b-1}$, for all $x \in \mathbb{R}$.*

*(2) Let $a > 0$. The base $a$ exponential is differentiable on its domain and $(a^x)' = \ln(a) \cdot a^x$, for $x > 0$.*

*(3) Let $a > 0$. The base $a$ logarithm is differentiable and $(\log_a(x))' = \frac{1}{\ln(a) \cdot x}$.*

**Exercise 148**

Prove theorem 44.

---

**Exercise 149**

Let $f$ be a positive real function whose derivative is continuous. Calculate:

$$\int \frac{f'(x)}{f(x)} \cdot dx$$

---

**Exercise 150**

Calculate

$$\int \tan(x) \cdot dx$$

---

**Exercise 151**

Let $f$ be a positive real function whose derivative is continuous. Calculate:

$$\int f'(x) \cdot e^{f(x)} \cdot dx$$

---

**Exercise 152**

Using $\ln(x) = 1 \cdot \ln(x)$, use integration by parts to compute $\int \ln(x) dx$.

---

**Exercise 153**

(1) Differentiate $\ln(x)$.

(2) Differentiate $e^x$.

(3) Integrate $x \mapsto e^x$.

(4) Differentiate the function $x \mapsto \ln(\ln(x))$.

(5) Differentiate the function $x \mapsto \ln(x^a)$ (Note that $a$ is not the variable!)

(6) Differentiate the function $x \mapsto \ln(a^x)$.

(7) Differentiate $x \mapsto e^{x^2}$.

(8) Using the fact that $u = e^{ln(u)}$ (if $u > 0$) differentiate $x \mapsto a^x$ (for $a > 0$ and $x > 0$).

(9) Same idea: Differentiate the function $x \mapsto x^x$.

---

**Exercise 154**

Differentiate $\ln(|x|)$.

---

This proves the following extension:

**Theorem 45**
*The antiderivative of $\frac{1}{x}$ is $\ln(|x|) + K$ for some constant $K$.*

## Mean value of a function

The mean value is unambiguous when we consider $n$ points, where $n$ is a positive integer. We now show that defining the mean value of a continuous function on $[a; b]$ as

$$\frac{1}{b-a}\int_a^b f(x) \cdot dx$$

is a natural extension of this concept.

Consider a continuous function $f$ and the interval $[a; b]$. Context is $a, b$ and $f$. Let $N$ be a positive ultralarge integer. Let $dx = (b - a)/N$ and $x_i = a + i \cdot dx$, for $i = 1, \dots, N$. Then the mean value of the function can be approximated by the mean value of the $N$ points $f(x_i)$, $i = 0, \dots, N - 1$. But

$$\frac{\sum_{i=0}^{N-1} f(x_i)}{N} = \frac{dx}{b-a}\sum_{i=0}^{N-1} f(x_i) = \frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i) \cdot dx \simeq \frac{1}{b-a}\int_a^b f(x) \cdot dx,$$

since $f$ is continuous on $[a; b]$.

The mean is the part of this number which is observable i.e., the integral. We therefore define:

**Definition 33**
*The **mean value** of a function $f$ continuous on $[a; b]$ is*

$$\frac{1}{b-a}\int_a^b f(x) \cdot dx.$$

The mean value is a number $\mu$ such that the area under the curve is equal to $\mu \cdot (b - a)$, i.e., the height of a rectangle of basis $(b - a)$ whose (oriented) area is equal to the integral.

**Theorem 46**
*If $f$ is a function continuous on $[a;b]$, then there exists a point $c \in [a;b]$ such that $f(c)$ is the mean value of the function on $[a;b]$.*

Note that theorem 46 is a restatement of theorem 2 which is the mean value theorem, for the antiderivative of $f$. When we claim that there is a $c \in [a;b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \cdot dx,$$

we are in fact asserting that there is a $c \in [a;b]$ such that

$$f(c) \cdot (b-a) = \int_a^b f(x) \cdot dx = F(b) - F(a),$$

and as $F'(x) = f(x)$, we conclude that there is a $c \in [a;b]$ such that $F'(c) \cdot (b-a) = F(b) - F(a)$.

It is also a consequence of lemma 2.

**Exercise 155**
Calculate the mean value of $x \mapsto x^2$ on $[-4;4]$.

---

**Exercise 156**
Calculate the mean value of $x \mapsto x^3$ on $[-4;4]$.

---

**Exercise 157**
Let $f : x \mapsto x^2$ and the interval $[0;t]$. Find the value of $t$ such that the mean value of $f$ over the interval is equal to $\pi$.

---

**Exercise 158**
An object falling on earth satisfies the equation $d(t) = \frac{1}{2}gt^2$ where $g \approx 9.81[m/s^2]$, $t$ is the time in seconds and $d(t)$ is the vertical distance.

If an object falls for $10s$, what is its average distance from its initial point?

---

**Exercise 159**
An object falling on earth satisfies the equation $d(t) = \frac{1}{2}gt^2$ where $g \approx 9.81[m/s^2]$, $t$ is the time in seconds and $d(t)$ is the vertical distance.

If an object falls for $10s$, what is its average distance from its initial point?

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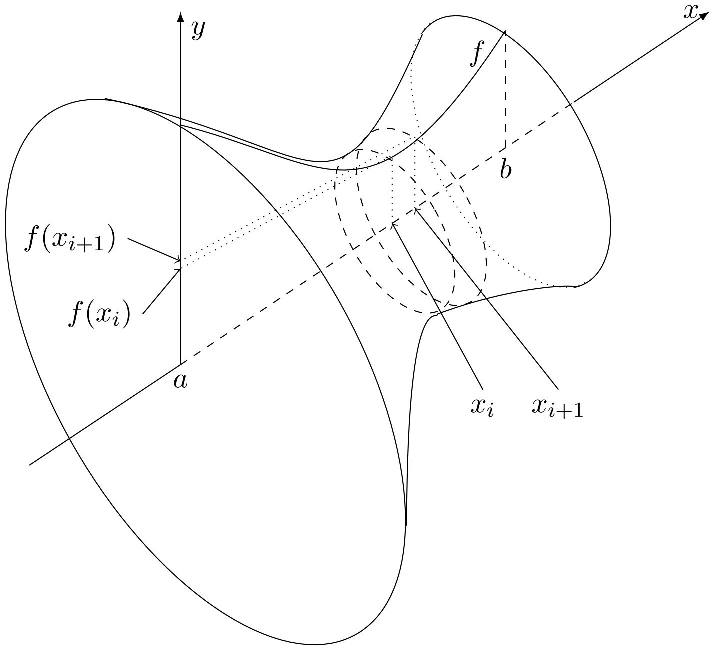
## Solid of Revolution




**Exercise 160**

An area is calculated by approximating the surface by ultrasmall rectangles. To find the formula for the volume of a solid of revolution, proceed in the same manner: consider that the solid is ultraclose to an ultralarge number of ultrathin disks. Find the formula for the volume of a solid of revolution given by a function $f$.

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**Exercise 161**

Evaluate the volume of the solid of revolution of $y = \frac{1}{x}$ around the $x$-axis between $x = 1$ and $x = 10$.

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**Exercise 162**

Evaluate the volume of the solid of revolution of $y = \frac{1}{x}$ around the $x$-axis between $x = 1$ and $x = +\infty$ i.e: take an ultralarge $N$ then show that the result does not depend on the choice of $N$.

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## Arc length

**Exercise 163**

Approximating the length of a curve by ultrasmall straight lines leads to the following definition. Explain why it is a reasonable definition (using the drawing).

**Definition 34**
*Let $f : [a; b] \to \mathbb{R}$ be smooth. Then the graph of $f$ has length*

$$L = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx.$$

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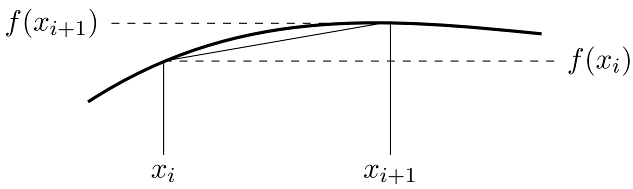


**Exercise 164**
Find the lengths of the following curves:

(1) $y = 2x^{3/2} \quad 0 \leq x \leq 1$

(2) $y = \dfrac{2}{3}(x+2)^{\frac{3}{2}} \quad 0 \leq x \leq 3$

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**Practice exercise 21** Answer page 96

Find the antiderivatives of the following functions:

- $f_a : x \mapsto 5x^4 - 2x + 4$
- $f_b : x \mapsto x^3 - 5x^2 + 3x - 2$
- $f_c : x \mapsto 2x - 1$
- $f_d : x \mapsto \dfrac{5}{4}x^4 - \dfrac{3}{4}x^2 + \dfrac{5}{2}x + \dfrac{3}{2}$
- $f_e : x \mapsto 2x + 1 - \dfrac{1}{x^2}$
- $f_f : x \mapsto 3 + \dfrac{2}{x^2} - \dfrac{5}{x^3}$
- $f_g : x \mapsto x^3 + \dfrac{1}{x^2}$
- $f_h : x \mapsto \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$
- $f_i : x \mapsto \dfrac{1}{\sqrt{x}} + \sqrt{x}$
- $f_j : x \mapsto (x+1)^2$
- $f_k : x \mapsto 15(3x-2)^4$
- $f_l : x \mapsto (2x+1)^3$
- $f_m : x \mapsto (3-x)^{11}$
- $f_n : x \mapsto (3-4x)^4$
- $f_o : x \mapsto \sqrt{3x-2}$
- $f_p : x \mapsto \dfrac{1}{\sqrt{x-1}}$
- $f_q : x \mapsto 4x(3-x^2)^5$
- $f_r : x \mapsto (2x-3)(x^2-3x+1)^4$
- $f_s : x \mapsto (3x^2-4x+1)(x^3-2x^2+x+3)^2$
- $f_t : x \mapsto (4x^2-5x)^2(16x-10)$
- $f_u : x \mapsto (3x-1)(3x^2-2x+5)^3$
- $f_v : x \mapsto \dfrac{2x}{(x^2+1)^2}$
- $f_w : x \mapsto \dfrac{2x+1}{(x^2+x+3)^2}$
- $f_x : x \mapsto x\sqrt{x^2+1}$
- $f_y : x \mapsto \dfrac{3x^2}{\sqrt{9+x^3}}$
- $f_z : x \mapsto (3x^2+1)\sqrt{x^3+x+2}$
- $f_A : x \mapsto e^{2x}$
- $f_B : x \mapsto \dfrac{1}{e^{3x}}$
- $f_C : x \mapsto xe^{-x^2}$
- $f_D : x \mapsto 2^{-x}$
- $f_E : x \mapsto e^{2x}\sqrt{1+e^{2x}}$
- $f_F : x \mapsto x^2e^x$
- $f_G : x \mapsto e^x \sin(x)$
- $f_H : x \mapsto \dfrac{e^x}{1+e^{2x}}$
- $f_I : x \mapsto \dfrac{1}{2x+3}$
- $f_J : x \mapsto \dfrac{2x}{x-1}$
- $f_K : x \mapsto \dfrac{x-1}{x+1}$
- $f_L : x \mapsto (\ln(x))^2$
- $f_M : x \mapsto \dfrac{\cos(x)}{1+\sin(x)}$
- $f_N : x \mapsto \ln(x)$
- $f_O : x \mapsto \dfrac{x}{x+1}$
- $f_P : x \mapsto \dfrac{1}{x\ln(x)}$

# 11
# Curve Sketching

**Practice exercise 22** Answer page 97
Sketch the following

- $g_1 : x \mapsto x\ln(x)$
- $g_2 : x \mapsto \dfrac{x}{\ln(x)}$
- $g_3 : x \mapsto \dfrac{e^x}{\ln(x)}$
- $g_4 : x \mapsto \dfrac{\sin(\sqrt{x})}{e^x}$
- $g_5 : x \mapsto \sin(\cos(x))$
- $g_6 : x \mapsto \cos(\sin(x))$
- $g_7 : x \mapsto \dfrac{e^x}{1+e^x}$
- $g_8 : x \mapsto \dfrac{1}{1+e^x}$
- $g_9 : x \mapsto \ln(x^2+1)$
- $g_{10} : x \mapsto \dfrac{e^x}{x-2}$
- $g_{11} : x \mapsto e^{-x^2}$
- $g_{12} : x \mapsto \dfrac{x \cdot e^x}{\ln(x)}$

## Answers to practice exercises

**Answers to practice exercice 19**, page 83

(1) $3$

(2) $\infty$

(3) $1/3$

(4) $1/\sqrt{3}$

(5) $\infty$

(6) $\infty$

(7) $-\infty$

(8) $\infty$

(9) $0$

(10) does not exist

(11) $\infty$

(12) does not exist

(13) does not exist

(14) $\infty$

(15) $0$

(16) $\infty$

(17) $0$

(18) $-3/2$

**Answers to practice exercice 20**, page 83

(1) $1/6$

(2) $1/16$

(3) $0$

(4) $-1/4$

(5) $1/2$

(6) $-1$

(7) $5$

(8) $\infty$

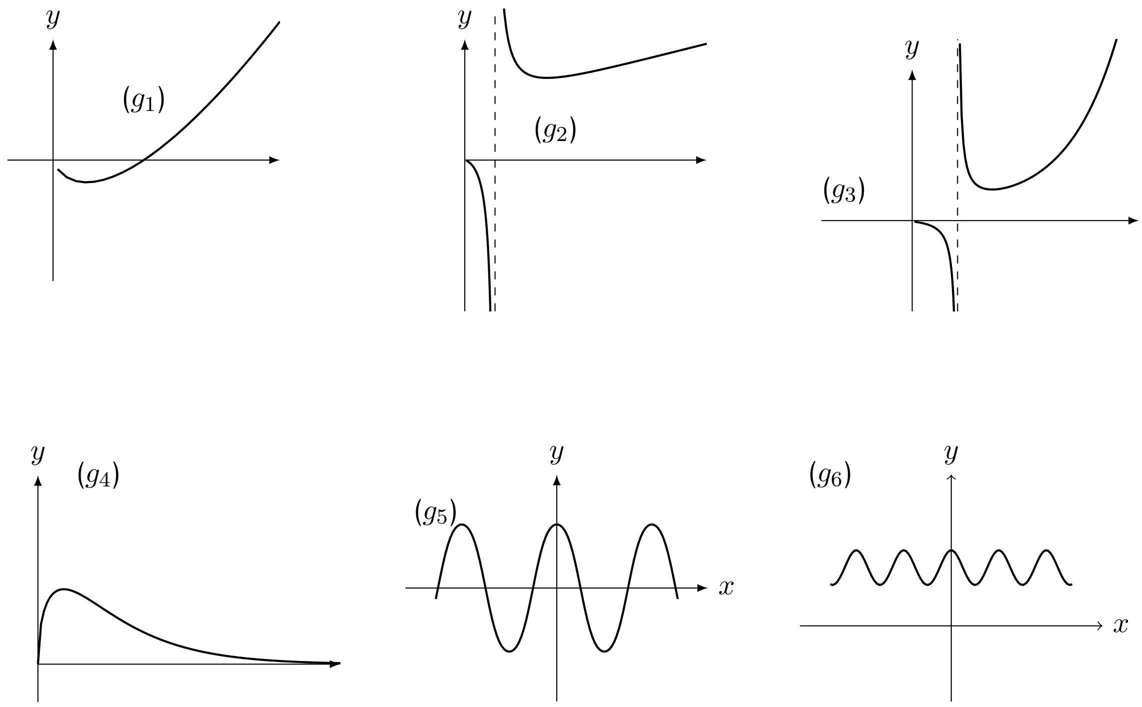
(9) $2$

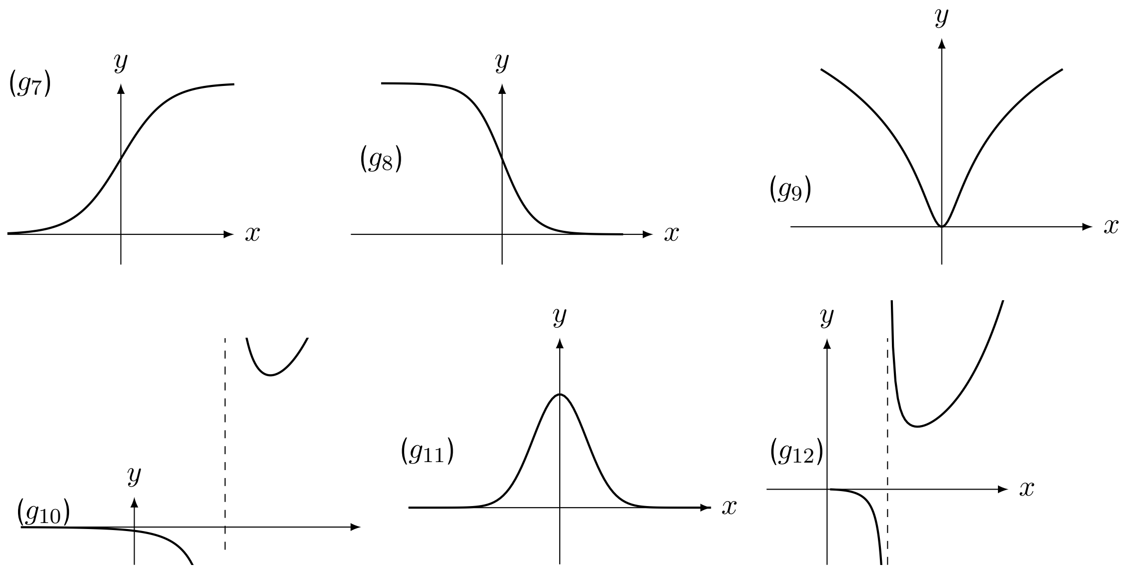
**Answers to practice exercice 21**, page 93
(Integration constant to be added)

- $F_a : x \mapsto x^5 - x^2 + 4x$
- $F_b : x \mapsto \frac{1}{4}x^4 - \frac{5}{3}x^3 + \frac{3}{2}x^2 - 2x$
- $F_c : x \mapsto x^2 - x$
- $F_d : x \mapsto \frac{1}{4}x^5 - \frac{1}{4}x^3 + \frac{5}{4}x^2 + \frac{3}{2}x$
- $F_e : x \mapsto x^2 + x + \frac{1}{x}$
- $F_f : x \mapsto 3x - \frac{2}{x} + \frac{5}{2x^2}$
- $F_g : x \mapsto \frac{x^4}{4} - \frac{1}{x}$
- $F_h : x \mapsto \frac{3}{4}\sqrt[3]{x^4} + \frac{3}{2}\sqrt[3]{x^2}$
- $F_i : x \mapsto 2\sqrt{x} + \frac{2}{3}\sqrt{x^3}$
- $F_j : x \mapsto \frac{1}{3}(x+1)^3$
- $F_k : x \mapsto (3x-2)^5$
- $F_l : x \mapsto \frac{1}{8}(2x+1)^4$
- $F_m : x \mapsto -\frac{1}{12}(3-x)^{12}$
- $F_n : x \mapsto -\frac{1}{20}(3-4x)^5$
- $F_o : x \mapsto \frac{2}{9}\sqrt{(3x-2)^3}$
- $F_p : x \mapsto 2\sqrt{x-1}$
- $F_q : x \mapsto -\frac{1}{3}(3-x^2)^6$
- $F_r : x \mapsto \frac{1}{5}(x^2-3x+1)^5$
- $F_s : x \mapsto \frac{1}{3}(x^3-2x^2+x-3)^3$
- $F_t : x \mapsto \frac{2}{3}(4x^2-5x)^3$

- $F_u : x \mapsto \dfrac{1}{8}(3x^2 - 2x + 5)^4$
- $F_v : x \mapsto -\dfrac{1}{x^2 + 1}$
- $F_w : x \mapsto -\dfrac{1}{x^2 + x + 3}$
- $F_x : x \mapsto \dfrac{1}{3}\sqrt{(x^2 + 1)^3}$
- $F_y : x \mapsto 2\sqrt{9 + x^3}$
- $F_z : x \mapsto \dfrac{2}{3}(x^3 + x + 2)\sqrt{x^3 + x + 2}$
- $F_A : x \mapsto \dfrac{e^{2x}}{2}$
- $F_B : x \mapsto -\dfrac{1}{3e^{3x}}$
- $F_C : x \mapsto -\dfrac{e^{-x^2}}{2}$
- $F_D : x \mapsto -\dfrac{1}{\ln(2)}2^{-x}$
- $F_E : x \mapsto \dfrac{1}{3}(e^{2x} + 1)^{\frac{3}{2}}$
- $F_F : x \mapsto e^x(x^2 - 2x + 2)$
- $F_G : x \mapsto \dfrac{e^x}{2}(\sin(x) - \cos(x))$
- $F_H : x \mapsto \arctan(e^x) - \dfrac{\pi}{2}$
- $F_I : x \mapsto \dfrac{\ln(x + \frac{3}{2})}{2}$
- $F_J : x \mapsto 2x + 2\ln(x - 1)$
- $F_K : x \mapsto x - 2\ln(x + 1)$
- $F_L : x \mapsto 2x\left(\dfrac{\ln(x)^2}{2} - \ln(x) + 1\right)$
- $F_M : x \mapsto \ln(\sin(x) + 1)$
- $F_N : x \mapsto x\ln(x) - x$
- $F_O : x \mapsto x - \ln(x + 1)$
- $F_P : x \mapsto \ln(\ln(x))$

**Answers to practice exercice 22**, page 95

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