Analysis

using ultrasmall numbers

Infinity itself looks flat and uninteresting. [...] The chamber [...] was anything but infinite, it was just very very very big, so big that it gave the impression of infinity far better than infinity itself.
(Douglas Adams: The Hitchhiker's Guide to the Galaxy)

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Preface for Instructors

Trained mathematicians all understand the following definition of continuity

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x) \quad |x - a| \leq \delta \Rightarrow |f(x) - f(a)| \leq \varepsilon \tag{*}$$

and all trained instructors know that, at introductory level, it is very complicated and students' understanding is not easy. The pedagogical solutions range from hand waving to metaphors involving undetermined arbitrarily small quantities, or an unformalised concept of infinitesimal. Admittedly, they do lead to students' understanding – at the cost of losing training in rigour.

Another way to circumvent the difficulty is to change the approach and use a version of nonstandard analysis – this is the choice we have made. It leads to defining continuity, with the same level of rigour as above, by

$$(\forall x) \quad x \simeq a \Rightarrow f(x) \simeq f(a) \tag{**}$$

There is a difficulty for the instructor. They probably have learned mathematics the mainstream way; it also means that the definition of continuity given at the top of the page was well understood. Furthermore, they have been working in a context where infinitesimals are at most a metaphor.

Introducing quantities that are **ultrasmall** or **ultralarge** will require accepting the challenge of changing some fundamental intuitions.

We do not question the efficiency or the correctness of the mainstream way of doing mathematics. We are concerned with maintaining a good level of rigour while introducing analysis, leaving open the possibility to go further in the same direction or switching to the classical $\varepsilon - \delta$ methods if required once the concepts have been understood.

The traditional writing of a limit

$$\lim_{x \to a} f(x) = L$$

reads

$$f(x) \text{ tends to } L \text{ when } x \text{ tends to } a$$

but one of the pedagogical difficulties is that each part of the sentence is meaningless when considered alone; x cannot tend to a by itself. Similarly with "$f(x)$ tends to L". The concept must be grasped in its entirety or not understood at all[1]. This is a big piece to swallow!

At introductory level, showing the full ε-δ definition certainly does not make things easier especially with the dependency of δ on ε.

Many difficulties disappear or become more palatable when using the concept of ultrasmall numbers. In particular, the limit is defined in such a way that each part of the sentence in $(**)$ has a meaning on its own – hence didactic and intellectual steps are smaller.

Most mathematicians have an intuitive idea of infinitesimals. These mental representations are often used to explain the fundamental concepts before rigorous formalisations are given.

[1] Do we need to remind the reader that x never really moves towards a?

Recent work by Karel Hrbacek, based on Yves Péraire's research offers a new formalisation of these ideas, mathematically rigorous yet still reasonably close to intuitive ideas and with a lower level of technical complexity. One of the authors and Olivier Lessmann have adapted this work to Geneva high school level and several teachers have been using it for nearly twenty years.

There have been several previous attempts to use nonstandard analysis for teaching, by Keisler [4], Stroyan [16] or Robert [15] for instance. These approaches used different approaches but one limitation was common to all: if h is infinitely small (in a way clearly defined in each theory) the derivative of $f : x \mapsto x^2$ was easy at $x = 2$ but difficult at $x = 2 + h$. The approach used here does not have this drawback.

More can be read about this approach in the article by Hrbacek et al. [1] and the book "Analysis with ultrasmall numbers" [2]. A point worth mentioning is that the principles added here extend classical set theory in such a way that no contradiction can be derived from them. In addition, it means that if a statement is written which does not use the concept of ultracloseness, a statement which could be read in the usual mathematics without the concept of observability, then it is true in one of the approaches if and only if it is true in the other: we *are* talking about the same things. This is discussed in Hrbacek et al. [2] with a similar proof by Péraire [14].

Several presentations have been made in Italy by one of the authors in meetings by Analisi non standard per le scuole superiori.

Observability and ultrasmall numbers

Extra axioms are used which allow to make an extra distinction within the real numbers: observability. It is the descriptive power of our mathematical language which is increased. No new objects are added.

The intuitive approach is to consider that the interval $[0, 1]$ contains infinitely many numbers – in the usual sense of the word: there is an uncountable infinity of real numbers in that interval. So some of these numbers must be really extremely very very close to each other and their difference must be really very very small. What we will do here is give a rigorous meaning to this intuitions about being *extremely close*.

Then "ordinary numbers" (the ones defined without this new concept, the ones such as $1, 2, \sqrt{2}, e, \pi \ldots$) are observable but that there are extremely small numbers which are so tiny that they are not observable. And if such a tiny number h is added to 2, then $2 + h$ is less observable than 2.

The only new symbol is "$\simeq$" which reads "ultraclose": a difference which is ultrasmall or zero.

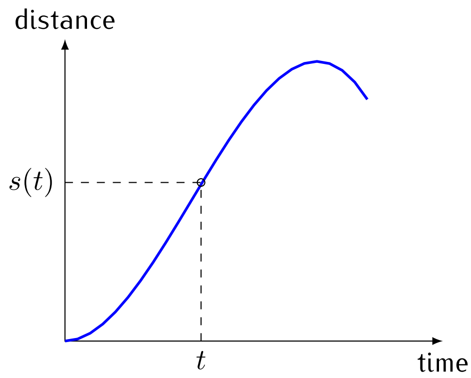
An ultrasmall number has the "flavour" of an infinitesimal.

⚠ The reason why we do not use the word "infinitesimal" is that the reciprocal of an infinitesimal would seem to be an infinitely large. But these distinctions are made <u>within</u> the real numbers, none of which are infinite in the classical way of defining the word. So in order to avoid confusion, we introduce the words ultrasmall and ultralarge, and the reciprocal of an ultrasmall is ultralarge. Obviously, there still is no reciprocal for 0.

Preface for Students

Calculus was developed independently by Isaac Newton (1642–1727) and Gottfried Wilhelm von Leibniz (1646–1716) in the last third of the seventeenth century as a general method for the study of changing quantities (functions). They approached the subject from different viewpoints. In order to understand the difference, let us look at a simple example of an important problem of calculus.

We consider a point-like object P moving in a straight line[2]. The position of P at time t is determined by the distance $s(t)$ of P from a fixed origin O.

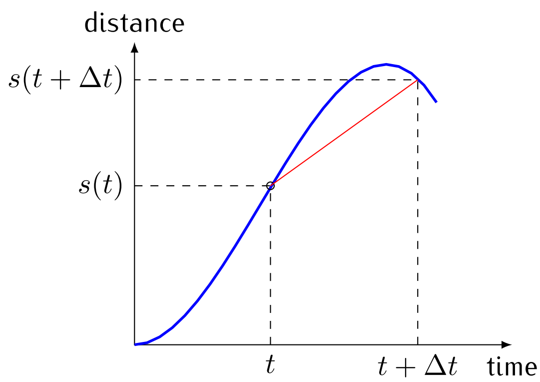


A fundamental assumption of mechanics is that the moving object has, at each time t, a definite instantaneous velocity $v(t)$, and one of its basic problems is to determine this instantaneous velocity, assuming that the distance function is known.

We begin by observing that the average velocity in an interval, say from t to $t + \Delta t$ where $\Delta t > 0$, can be obtained by a straightforward algebraic computation.

If $s(t)$ is the distance of the object from the origin at time t, $s(t+\Delta t)$ is its distance from the origin at time $t + \Delta t$, hence, during the time interval from t to $t + \Delta t$ the object has travelled the net distance Δs equal to $s(t + \Delta t) - s(t)$, with the average velocity

$$\frac{\Delta s}{\Delta t} = \frac{s(t + \Delta t) - s(t)}{\Delta t}. \tag{1}$$



[2]Note that the trajectory is a straight line, but this does not mean that the object moves at constant speed.

If we want to use something similar to equation 1 to define the instantaneous velocity, the question is about s at time t, and the quantity Δt would be just a *convenient temporary variable* and one of the goals here is to find a way to get a result which would be independent of the choice of the particular Δt.

As an instant has no measurable duration, one might think that the instantaneous velocity $v(t)$ at time t could be obtained from equation (1) by setting $\Delta t = 0$. However, this idea does not work because the resulting expression will contain a division by zero which is mathematically meaningless.

Newton considered the quantity Δt to vary until is was "vanishingly small" yet not zero. This led to a formalisation by Karl Weierstrass. In this approach, no "infinitesimal quantities" are considered.

The resulting theory has proven to be extremely fruitful but difficult to grasp at the introductory level.

Leibniz considered that Δt was an infinitesimal quantity. It was only in 1960 that Abraham Robinson (1918–1974) showed how to work rigorously with infinitesimal quantities. It was followed by Edward Nelson, Petr Vopěnka, Karel Hrbacek and others.

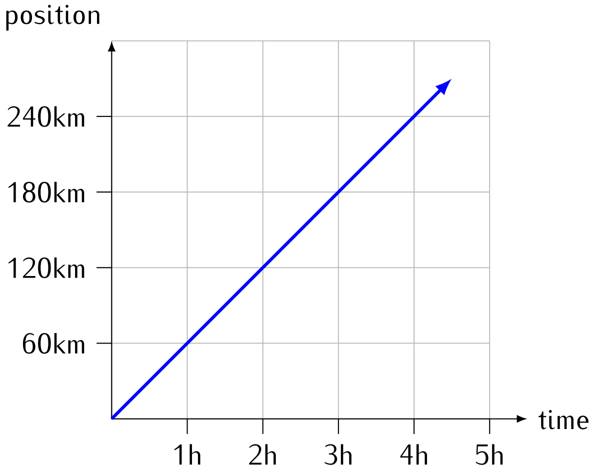
The approach used in this book is based on a similar concept that uses **ultrasmall** quantities and has been developed with the aim of simplifying the learning of calculus yet remaining rigorous.

Some exercises have worked out solutions and are named **Practice Exercise** followed by a number and a reference to the page where the answer is.

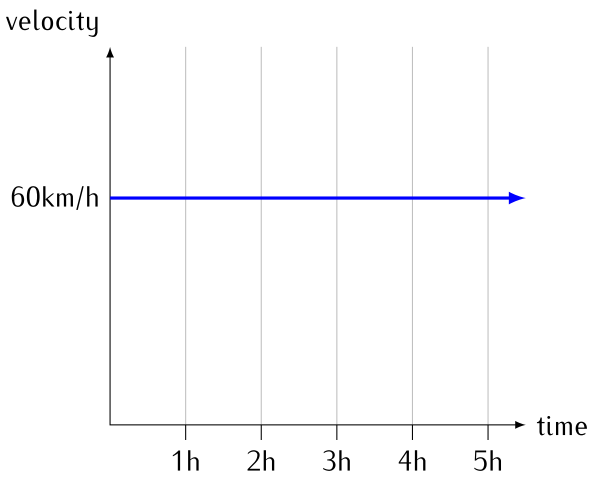
Prologue

Velocity and position

Suppose the velocity[3] of a car is constant and equal to 60km/h. The function[4] which describes the position of the car along its trajectory is a straight line.



Since the velocity is constant, the function which describes the velocity is a horizontal line.



[3]Velocity is speed with a direction. Speed is always positive (or zero); velocity can be negative.

[4]Function: relation between an input set (usually represented by variables x or t) and an output set (usually represented by y) such that to each and every input corresponds exactly one output.

The vertical units are different in each graph so there is no requirement that the grids correspond.

The input (horizontal) is time, so the unit can be hours. The output (vertical) for position can be kilometres. The output for velocity can be kilometres per hour.

On the position graph, the velocity is given by the slope $\frac{\text{vertical}}{\text{horizontal}}$, which in units is $\frac{km}{h}$

On the velocity graph, the variation of position is given by velocity×time, which in units is $\frac{km}{h} \times h = km$. And this is an **area**![5]

> ⚠ Note the difference: velocity (deduced from position) is local. It is possible to give the velocity at any given time. Position (deduced from velocity) is global. It is only possible to find the variation of the position over an interval of time.

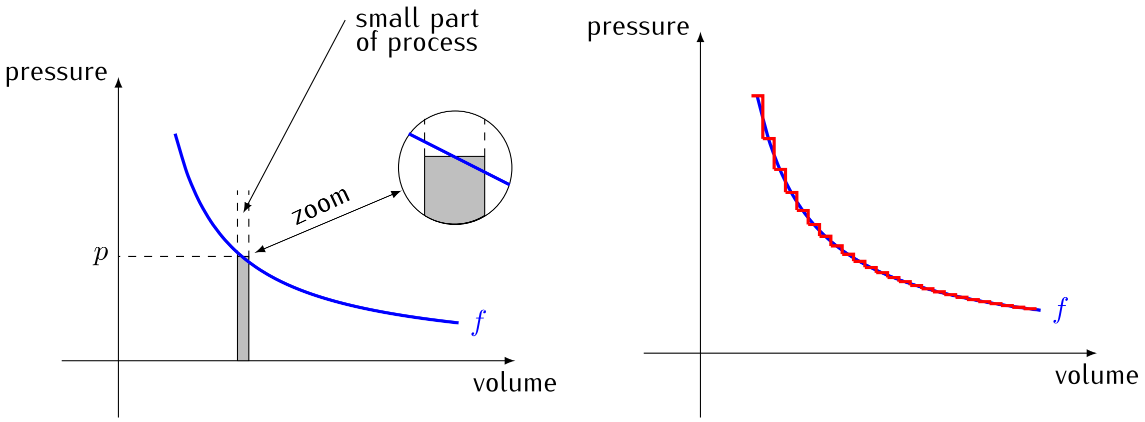
Processes

Suppose you have a certain amount of gas, at high pressure, enclosed in a container that has a moving wall. Intuitively, the pressure of the gas is able to displace away the moving wall and, in doing so, the chamber increases its volume and the gas is less compressed. Suppose that you need to calculate the product between the gas pressure and the volume increase produced by the pressure. Indeed, the container in our example is very similar to the combustion chamber of an internal combustion engine (those of the kind you might find under the hood of your car, if it is not electric of course) and the product between the gas pressure and the volume change that it produces is the mechanical work (roughly speaking, the energy) that you can extract from the engine. Now, while the moving wall is displacing, the chamber volume is increasing, and the gas pressure inside the chamber does not remain constant; it decreases. The gas inside the container is undergoing a *thermodynamic process*. An important question immediately arises: what value of the gas pressure should we take to calculate the mechanical work: the initial pressure, the final pressure, or some pressure in between the initial and the final stage of the process? If we consider the entire process as a whole, the question is relevant. However, the question becomes less relevant if we break the whole process into tiny pieces and consider just one of such pieces. During that small part of the process, the gas pressure will change, but the change will be small.

To calculate the mechanical work during the small piece of process, we will take the (small) volume variation produced by the gas and we will multiply it by the mean pressure during the piece of process. This is equivalent to assuming that the gas pressure remains constant during the small process. Indeed, we commit a slight error in such assumption.

We have now been able to calculate the mechanical work done during a small piece of the entire process. To calculate the overall work done during the whole process, we just need to sum all the work done during all the pieces.

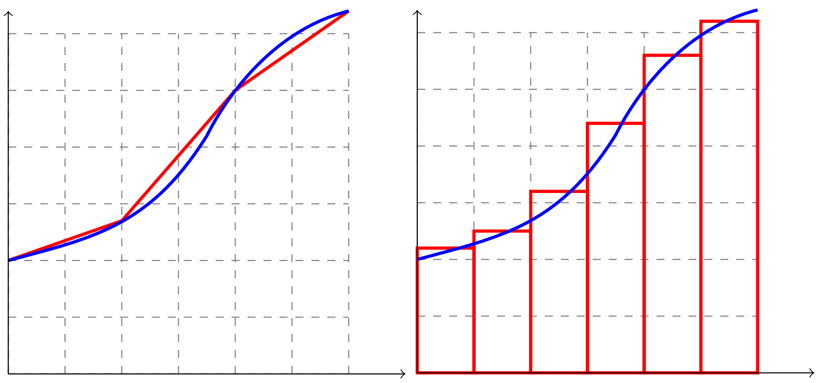
[5] The area is a measure, whereas the surface is a physical object. The unit for an area on a graph is the product of horizontal unit by vertical unit. The resulting unit can very well be a distance.



The start of the journey

What if the function is not a straight line?

A curve can be approximated by a piecewise linear function whose slope is easily calculated by pieces. It can also be approximated by a "staircase" function whose area is calculated by adding the areas of the rectangles.



But how much information is lost in the approximation process?

The main goal of the subject called **mathematical analysis** is to check when and how to approximate a curve by pieces of straight lines and when and how to approximate areas by rectangles and to understand what these can be used to calculate.

Intuitively, it should seem clear that in order for the approximation to be good, the pieces of straight lines or the rectangles must be small – or that the number of pieces is large – so that the straight lines are never far from the curve. The crucial questions are: How small? and How large? How close?

As mentioned page viii, for the slope, the question is about the function and the point at which the slope is to be found; for the area, the question is about the function and about the interval over which the area is to be found. All other variables are convenient temporary variables and the situation is not really about these. Furthermore, results should be independent of their actual value.

Chapter 1

Numbers and functions

Natural numbers

The first numbers encountered by children are counting numbers.

In mathematics, we start by **whole numbers** or **natural numbers**.

They differ from counting numbers by starting with 0.

The set (or collection) of these numbers is symbolised by $\mathbb{N}$. It is an infinite set and can be indicated by giving (between curly braces) its first elements and continuing with dots.

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

To indicate that a number is a natural number, the $\in$ symbol is used.

$$0 \in \mathbb{N}$$

This reads as "zero is in N" or "zero belongs to N".

These numbers have some specific properties:

- Every set of natural numbers has a smallest element
- If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

  (in the natural numbers, there is a "next number")
- If a and b are natural numbers, then $a + b \in \mathbb{N}$ and $a \cdot b \in \mathbb{N}$
- If a and b are natural numbers, $a - b$ and $a : b$ are not necessarily natural numbers. $5 - 3 \in \mathbb{N}$ but $4 - 7 \notin \mathbb{N}$, similarly $12 : 4 \in \mathbb{N}$ but $12 : 5 \notin \mathbb{N}$

  This is the **closure** of $\mathbb{N}$ under addition and under multiplication, meaning that the result is of the same kind of numbers.
  The natural numbers are not closed under division or subtraction, because the result of a subtraction or of a division is not necessarily a natural number.

Integers

To allow for closure under subtraction, the set of **Integers** is conceived. Integers are obtained by extending the natural numbers to the negative side.

$$\mathbb{Z} = \{0, +1, -1, +2, -2, +3, -3, +4, -4 \dots \}$$

Note that the number $2 \in \mathbb{N}$ is the same number as $+2 \in \mathbb{Z}$. The "plus" sign is unnecessary when the only numbers used are positive, but it can be useful to clarify when both negative and positive numbers are used. When no sign is in front of a number, it is assumed to be positive.

These numbers have some specific properties:

- not every set of integers has a smallest element: it can extend to infinity on the positive and also on the negative side.
- If $n \in \mathbb{Z}$, then $n + 1 \in \mathbb{Z}$ and $n - 1 \in \mathbb{Z}$

  (in the integers, there is a "next number" and there is a "previous number")
- If a and b are integers, then $a + b \in \mathbb{Z}$, $a - b \in \mathbb{Z}$ and $a \cdot b \in \mathbb{N}$

  This is the **closure** of $\mathbb{Z}$ under addition, subtraction and under multiplication.
  The integers are not closed under division.
- If n is an integer, then $-n$ is an integer (note that since n can be positive or negative, one must not assume that $-n$ is negative).

Infinity

We have used the expression "the set is infinite" and "it can extend to infinity." We clarify here the meaning.

We say that a set is infinite if there is no natural number which can be used to say how many elements there are in the set. Here, we are concerned with counting the number of elements in the set.

Example: the set of all even natural numbers is infinite.

Proof: Assume n is the number of even numbers, this means that $2n$ is the greatest even number. But $2n + 2$ is also an even number so there would be $n + 1$ even numbers. Which contradicts the assumption that there is a number that can count the even numbers.

If we write that a set of natural numbers *extends* to infinity, we mean that there is no natural number which is bigger than all numbers in the set. Here, we are concerned with identifying the biggest element in the set.

For example, the set of even numbers extends to infinity, since no number can be written which is bigger than all the other even numbers.

In other words, this means that no natural number *is* in itself infinite, since no natural number can be bigger than all natural numbers including itself.

For negative integers a similar reasoning applies: no negative number can be smaller than all negative integers so there is no negative infinite integer.

These two sets are infinite sets of finite numbers.

The symbol for "extends to infinity" is ∞. If a set extends to infinity on the negative side, the symbol $-\infty$ is used.

⚠ The infinity symbol (∞) is not the symbol of a number and cannot be used in operations, other than indicating that it is a negative infinity by adding the minus sign in front: $-\infty$.

All numbers in $\mathbb{N}$ and $\mathbb{Z}$ are finite yet there are infinitely many of them...

Rational numbers

To extend closure to the division operation, rational numbers are conceived. Fractions are introduced to characterise rational numbers, symbolised by $\mathbb{Q}$.

$$\mathbb{Q} = \{0,\ 1,\ -1,\ \frac{1}{2},\ -5,\ \frac{3}{2},\ -3,\ 7,\ -\frac{41}{17}\ \dots\}$$

These numbers have some specific properties:

- If a and b are rationals (written $a, b \in \mathbb{Q}$), then $a + b \in \mathbb{Q}$, $a - b \in \mathbb{Q}$, $a \cdot b \in \mathbb{Q}$, and, if $b \neq 0$, $\frac{a}{b} \in \mathbb{Q}$.

  This is the **closure** of $\mathbb{Q}$ under addition, subtraction multiplication and division (except division by zero).

- If n and m are in $\mathbb{Q}$, then $\frac{n+m}{2}$ is in $\mathbb{Q}$ and is between n and m. (This is called **density**.) (Between any two rational numbers is another rational number – hence infinitely many rational numbers.)

  ⚠ As a consequence: there is no immediate next number and no immediate previous number in the rationals.

- The decimal expression of a rational number is either a decimal number of finite length or eventually has a repeating sequence of digits after the decimal point.

Real numbers

Practice exercise PE1 Answer page 119

Prove that $\sqrt{2} \notin \mathbb{Q}$

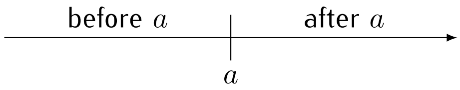
Hint: assume there is a fraction in simple form equal to $\sqrt{2}$ (therefore that its square is equal to two) and try to deduce that in fact it is not in simple form, hence assuming there was a fraction equal to $\sqrt{2}$ is contradictory.

To allow closure under the square root operation, real numbers are conceived. A number whose decimal expansion does not end nor repeat is not a rational number. Examples: $0.12345678910111213\dots$, $1.101001000100001\dots$ are not rational numbers. These are real numbers.

Real numbers are **closed** under addition, subtraction, multiplication, division and roots and are symbolised by $\mathbb{R}$.

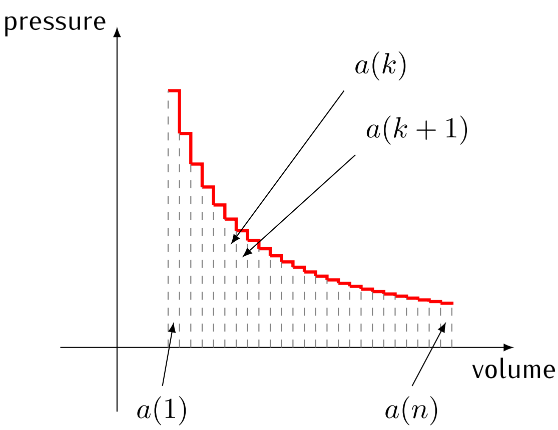
Another important property of the set of real numbers is that it is a dense set (see page 7). This means that there is no immediate predecessor or immediate successor: there is no next number after, say, 2. It is not possible to go from 2 to 3 by going from one number to the next. Yet even though there is no immediate successor or predecessor, numbers which are smaller than a given number, will be said to be "before" that number, and those which are greater are "after" that number.

When drawing the **real number line**, number which are before are on the left, and those that are after are on the right. The line is **oriented**.



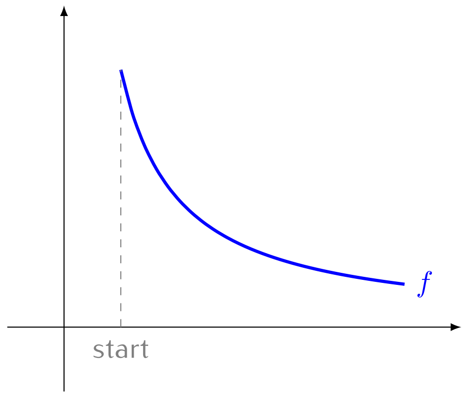
About successors

Whether a set of numbers satisfies the successor property or not is crucial for analysis. Imagine the area under a staircase function (see page 3):



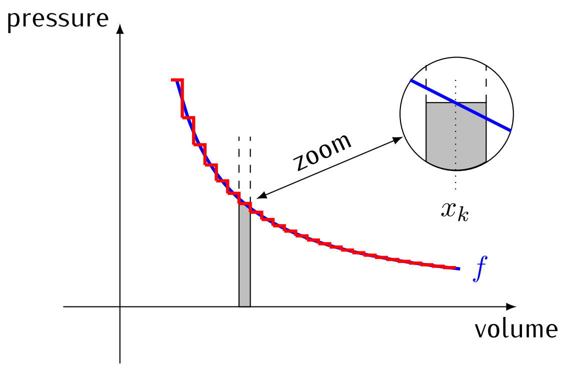
It is possible to calculate the area by calculating the area of each rectangle, written as $a(1), a(2), \ldots a(k), a(k+1), ..., a(n)$ and add all of these by adding them *one after the other* since that is the way addition is performed.
This is a crucial point; it is not possible to add all numbers in a series in one step. Addition is defined between *two* numbers: we add the first two, then the third to this result and so on, until the last one ($a(n)$).

Now imagine we want to calculate the area under a curve given by a function f. If we wanted to add the area under *all* the points, the starting point is clear but there are two immediate big problems.



The first problem is that the first area to add seems to be zero (the area under a single point, the area of a line, which according to geometry has no width). The second problem is that even if we solve the first problem, there is no *next* value to add: there is no successor for the addition.

Approximating the curve by a staircase function does not immediately solve the problem:



If we approximate the curve by a staircase, on the part in the zoom, there seems to be a more or less triangular part missing on the top left and a bit too much on the top right. Since we do not yet know how to calculate the exact area under the curve, we cannot be certain that they compensate exactly. But we can add one rectangle to the next one.

Hence dividing the area into slices, enables to calculate each slice and sum up the pieces by adding them *one after the other*. The area of a single slice is given by height times width: $f(x_k)$ times "the width of the slice", where x_k is the midpoint of the slice.

As mentioned on page 3, one of the main goals of analysis is to go from approximation to exact result and this is done by slicing dense sets into pieces where the successor relation holds – and then some extra work is still needed...

Intervals of real numbers

An interval of real numbers is the set of all real numbers between two bounds. In the following, "all numbers" means "all real numbers".

The following cases may occur, for a and b both in $\mathbb{R}$

- $[a, b]$: all numbers between a and b, including a and b (closed interval)

  Alternative notation $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ which reads "all x in $\mathbb{R}$ such that a is less than or equal to x which is less than or equal to b".

- $[a, b[$: all numbers between a and b, including a excluding b (interval closed on the left, open on the right)

  Alternative notation $\{x \in \mathbb{R} \mid a \leq x < b\}$

- $]a, b]$: all numbers between a and b, excluding a including b (interval open on the left, closed on the right)

  Alternative notation $\{x \in \mathbb{R} \mid a < x \leq b\}$

- $]a, b[$: all numbers between a and b, excluding a and b (open interval)

  Alternative notation $\{x \in \mathbb{R} \mid a < x < b\}$

- $[a, \infty[$: all numbers greater than a, including a

  Alternative notation $\{x \in \mathbb{R} \mid a \leq x\}$

- $]a, \infty[$: all numbers greater than a, excluding a

  Alternative notation $\{x \in \mathbb{R} \mid a < x\}$

- $]-\infty, b]$: all numbers less than a, including a

  Alternative notation $\{x \in \mathbb{R} \mid x \leq b\}$

- $]-\infty, b[$: all numbers less than a, excluding a

  Alternative notation $\{x \in \mathbb{R} \mid x < b\}$

- $]-\infty, \infty[$: all real numbers

The number on the left of the interval notation is not larger than the number on the right.

Note that an interval is always open on the side of the infinity symbol, since infinity cannot be reached...[1]

An interval of real numbers contains either zero elements (denoted by $\emptyset$), one element or infinitely many elements. Because of the density property, as soon as there are two distinct elements there are infinitely many elements in between.

$]1, 1[= \emptyset$: an empty set.

$[1, 1] = \{1\}$: a set containing only one element.

$[0, 1]$: a set that contains infinitely many elements.

Tiny and huge

If an interval such as $[0, 1]$ contains infinitely many numbers, the distance between some of them must by tiny – in some sense to be specified. We explore intuitively some of the properties that these tiny numbers can have.

Other names are "infinitely small" and "ultrasmall". Some of the properties may differ according to how they are mathematically defined, but we concentrate here on intuition. In section 2, page 17, a more rigorous formalisation is given.

[1] For the meaning of the ∞ symbol, reread caution note on page 7.)

When working with the concept of tiny numbers, some fuzziness is to be expected: some results will simply state that two numbers are extremely close: their difference is tiny – without possibility of being more specific.

Practice exercise PE2 Answer page 119

If δ is a positive value which is really tiny (even tinier than that!),

(1) what can you say about the size of δ^2, $2 \cdot \delta$ and $-\delta$?

(2) what can you say about $2 + \delta$ and $2 - \delta$?

(3) what can you say about $\frac{1}{\delta}$?

Practice exercise PE3 Answer page 120

If N is a positive huge number (really very huge!),

(1) what can you say about N^2, $2N$ and $-N$?

(2) what can you say about $N + 2$ and $N - 2$?

(3) what can you say about $\frac{1}{N}$?

(4) what can you say about $\frac{N}{2}$?

Practice exercise PE4 Answer page 120

Let $a \simeq b$ stand for $a - b$ is tiny, with $0 < a < b$. We assume here that a is not tiny (hence neither is b)

(1) what can you say about $\dfrac{a}{a-b}$?

(2) what can you say about $\dfrac{a}{b-a}$?

(3) what can you say about $\dfrac{b-a}{a}$?

(4) what can you say about $\dfrac{a-b}{a}$?

(5) what can you say about $\dfrac{a-b}{b-a}$?

Functions

Assume you want to bake a cake whose recipe requires one egg. Once you have the flour, the sugar and whatever is required in the recipe, you look up in the refrigerator and see that you have six eggs. Can you make your cake?
Yes of course! because if you have six eggs, then you have one egg.
On the other hand, if the recipe specifies one egg, you will not use all six, because in a recipe, one means "exactly one".
In mathematical recipes, the terminology will be either "one" (meaning "at least one") or "exactly one". (Sometimes, instead of "exactly one", the phrase "one and only one" will be used).
If the recipe states to add cream on "each" slice it is the same thing as stating to add cream on "every" slice.

Definition 1 (Function) A function f is a relation between two sets (input set A and output set B) such that for *each* element x of the input set there is *one and only one* corresponding element $f(x)$ of the output set.
Notation:

$$\begin{aligned} f : A &\to B \\ x &\mapsto f(x) \end{aligned}$$

The input set is also called the **domain** and the output set is called the **range** or **codomain**.

Practice exercise PE5 Answer page 120
Describe relations between two sets which are not functions.

The function is usually given by a mathematical rule. The rule can be given by a collection of rules, provided they respect that for any given input, exactly one of the rules apply.

When nothing is specified, the domain of the function is assumed to be the biggest possible input set for the specifying rule. (For values not in the domain, for some reason, the output is not defined, such as division by zero or square roots of negative numbers.)

Real functions

Functions whose domains are sets of real numbers and whose ranges are sets of real numbers are said to be **real functions**. In all this study of analysis, functions are assumed to be real functions so it will not be specified.

For graphical representations, the input set is usually established as a horizontal line in x (or t) and the output set is established as a vertical line in y. The positive directions are to the right and upwards.

Exercise 1

Using any available method, sketch the graphs of the following functions.

(The indication of $\mathbb{R}$ as output simply means that you must determine the biggest and smallest values to make your drawing and that there is no other restriction on the output.)

(1)
$$\begin{aligned} f: \quad [-3,3] &\to \mathbb{R} \\ x &\mapsto 2x+1 \end{aligned}$$

(2)
$$\begin{aligned} f: \quad [-3,5] &\to \mathbb{R} \\ x &\mapsto -2x-3 \end{aligned}$$

(3)
$$\begin{aligned} f: \quad [-3,3] &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

(4)
$$\begin{aligned} f: \quad [-2,2] &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$

(5)
$$\begin{aligned} f: \quad [-4,4] &\to \mathbb{R} \\ x &\mapsto |x| \end{aligned}$$

(6)
$$\begin{aligned} f: \quad [-4,4] &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \end{aligned}$$

Comment *This last function is defined* ***by parts*** *and could seem to be the concatenation of two functions. One must understand that it* <u>*is*</u> *a single function since for any x, exactly one of the rules apply. (Nothing in the definition of a function stipulates that the same rule must apply to all x. So the general rule here is divided into two subrules.)*

Exercise 2

If $f(x) = x^2 + 1$ and $g(x) = x + 2$ are two rules for functions, give the following results in simple form.

(1) $f(1)$

(2) $f(-2)$

(3) $f(3+a)$

(4) $f(3x)$

(5) $3f(x)$

(6) $f(g(2))$

(7) $g(f(2))$

With the intuitive concept of tiny:

Practice exercise PE6 Answer page 121

Let $f : x \mapsto x^2$, and let δ be tiny and positive.

(1) Draw the result of a zoom centred on the point $\langle 1, 1\rangle$ of f so that δ becomes visible.

Show, on the drawing, the values 1 and $f(1)$; $1+\delta$ and $f(1+\delta)$; $1-\delta$ and $f(1-\delta)$

What does the curve look like?

(2) Similar question for a zoom centred on $\langle -1, 1\rangle$

Practice exercise PE7 Answer page 122

Let $g : x \mapsto |x|$, and let δ be tiny
Draw the result of a zoom centred on $\langle 0, 0\rangle$ of g so that δ becomes visible.
Show, on the drawing, the values 0 and $g(0)$; δ and $g(\delta)$; $-\delta$ and $g(-\delta)$.

Practice exercise PE8 Answer page 123

Draw the following function for x between -2δ and 2δ but with a vertical scale between -1 and 1. This means a different scale for the vertical axis than for the horizontal axis.

$$k : x \mapsto \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Summary of this chapter

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *are sets of numbers: natural numbers, integers, rational numbers, real numbers. We will concentrate on functions from real numbers to real numbers.*

Closure of an operation in a number set means that if the inputs are two numbers from that set, the result is a number in that same set.

Intuitively: tiny numbers reproduce the same structure at a smaller level of observation. When we zoom in on a line, we gain information (we can distinguish numbers which seemed blurred into one).

When zooming on a curve, it may seem to become indistinguishable from a straight line. In other cases, a pointed vertex may appear and remain pointed after zooming in, or a step may remain a step.

If the zoom is indistinguishable from a straight line, the slope of that line can be determined.

infinity

The ancient Greeks considered two sorts of "infinities".

- A **potential infinity**: counting never stops. In symbols,

  $$\forall x\ \exists y \quad y > x$$

  which could be read: "for any value I can give, you can find one which is greater"

  That numbers in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ satisfy this property is beyond doubt.

- An **actual infinity**: there is a quantity which is bigger than any number you can count. In symbols

  $$\exists y\ \forall x \quad y > x$$

  It follows that if the domain of x real numbers, then y itself cannot be a real number since it would have to be greater than itself!

  It may be harder to conceive but this infinity is the one that measures the size of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, it is the measure of the number of pints on a line.

In all cases, infinities are **not** numbers in the usual sense, which is why the symbol "∞" may not be used in operations..

When one considers that there are infinitely many real numbers in the interval $[0, 1]$, it is an actual infinity.

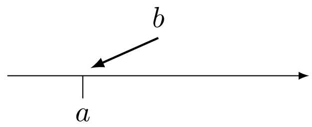
Chapter 2

Principles of proximity: formalisation

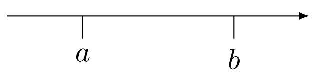
There are infinitely many (real) numbers in the interval $[0, 1]$, but then some of them must be really very close to each other.

Some are so close that without a microscope, their difference is not **observable**.

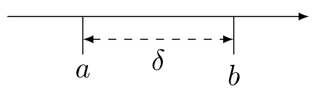
Assume a is observable.
If b is so close to a that the difference cannot be observed, we consider that b is not observable (in the sense that it cannot be individually distinguished from a.)



When we zoom in and the difference between a and b becomes observable, a remains observable.



The (overall) observability is now that of b, and if we write $\delta = b - a$ then δ is so small it could not be observed before the zoom.



So far, you probably never thought of using a microscope to look at numbers, so the numbers you already know are **always observable** (with or without microscope).

(1) If we do not need a microscope to see a number, we say that it is **always observable**. Tiny numbers are not as observable. But we can zoom in to see them.

(2) Also, if we can see the number 2, for example, (not the interval from zero to two, just the point which has value two) when we zoom in, we can distinguish 2 from $2+\delta$, but *(important fact!)* we can still see 2. Observability does not get lost when zooming in; observability does not get lost when tiny numbers are also made observable.

(3) Now imagine that we have zoomed in, so we can see $2+\delta$; it is possible to imagine numbers yet tinier, which would need a much more powerful microscope to be distinguished. This means that when we specify that a number is tiny, we must be able to specify "compared to which numbers", that is to say, according to our metaphor, we need to specify what power is needed from the microscope to observe this number. We will refer to this as the "observability"

(4) If two numbers are observable relatively to a given value (they are as observable as that value) then their product, difference, quotient and results of other operations are also observable relatively to that given value. This is **closure** of observability in a similar sense as mentioned on page 5 for addition and multiplication of natural numbers.

Tiny numbers do not appear in a result if they have not been in the input.

Since "tiny" is a familiar word which can have ambiguous meanings, we choose another word: **ultrasmall**.

Formalisation

Consider a problem to be solved, such as calculating the slope of a function f at a given point a or calculating the area under a curve between two points b and c, the question is about f and a, or about f, b and c. Convenient temporary variables may occur but the problem is not about these.

Observability

(1) Numbers defined without reference to observability are always observable.

(2) There are numbers which are not always observable.

(3) Observability is determined by the numbers and parameters the problem is about.

(4) The result of operations between observable numbers are observable. This is **closure of observability**, and will be referred to simply as "closure".

Numbers defined without reference to observability are always observable.
This means that all the numbers you have encountered so far, such as

$$1 \qquad 0 \qquad 1000000 \qquad -5 \qquad \sqrt{2} \qquad \pi$$

are always observable. Other numbers such as ultrasmall numbers are less observable.

If a is always observable and b is not always observable, then when we zoom in to observe b, a remains observable. So $a+b$ is as observable as b.

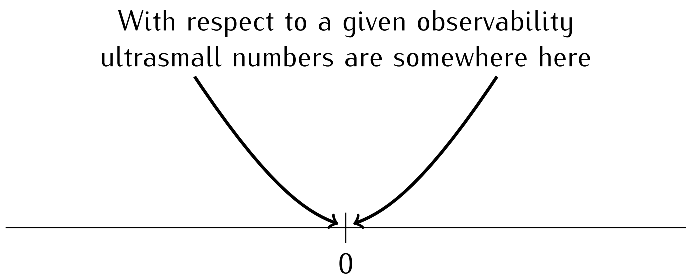
"Observable" always refers to a predetermined situation. No number has any form of observability alone, except maybe to say that it is as observable as itself.

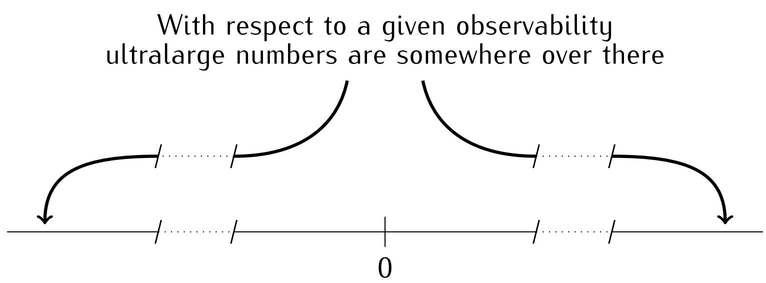
Definition 2 (Ultrasmall) If a non zero number is smaller in absolute value than any non zero positive observable number, it is **ultrasmall**.

Ultrasmall
Whatever the observability, there are numbers which are ultrasmall.

Definition 3 (Ultralarge) A real number is **ultralarge** if it is larger in absolute value than any positive observable number.

Ultralarge numbers can be positive or negative. If we want to specify, we may use the terminology "ultralarge positive" for instance. If we indicate simply "ultralarge" it is similar to when we specify "ultrasmall" which can be either positive or negative (but obviously not both at the same time...)





⚠ Ultralarge numbers are not "infinity", they are real (huge) numbers and as such can be used in operations.

Definition 4 (Ultraclose) Let a, b be real numbers. We say that a is **ultraclose** to b, written

$$a \simeq b$$

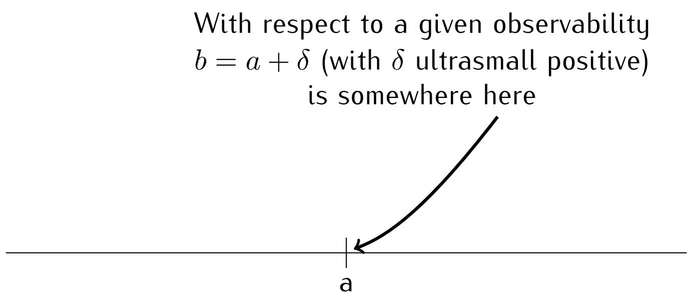
if $b - a$ is ultrasmall or if $a = b$.

Since there is a reference to ultraproximity, it is understood that it refers to some observability.

In the example above, there is an ultrasmall value δ such that $b = a + \delta \simeq a$

In particular, $\delta \simeq 0$ if δ is ultrasmall or zero.

It is important for mathematical practice to be able to distinguish between being *very close to zero but not equal to zero* or being *very close to zero and possibly equal to zero*. See property 3, page 23, property 4, page 23 and property 2, page 20 for first examples of this distinction.



Relative to observability:

The concepts of **observability**, **ultrasmall**, ultralarge and **ultraclose** are always relative to all the values and parameters of the situation, hence it is not necessary to specify them explicitly.

The following property is fundamental, because everything that follows is a consequence of it. It is extremely important to fully understand the claim.

Property 1 Let ε be ultrasmall and let a be observable and not zero. Then

$$a \cdot \varepsilon \quad \text{is ultrasmall}$$

Proof

By contradiction. Assume that $a \cdot \varepsilon$ is not ultrasmall: written $a \cdot \varepsilon \not\simeq 0$.

If this were smaller, in absolute value, than all non zero positive observable numbers, it would be ultrasmall, hence since it is assumed not ultrasmall, there is an observable number between $a \cdot \varepsilon$ and zero. And since the definition mentions absolute values, we have that there is an observable strictly positive b such that $|a \cdot \varepsilon| \geq b > 0$. But $|a \cdot \varepsilon| = |a| \cdot |\varepsilon|$. So $|a| \cdot |\varepsilon| > b > 0$.

Then $|\varepsilon| \geq \frac{b}{|a|} > 0$. By closure $\frac{b}{|a|}$ is observable. This contradicts that ε is ultrasmall. □

Property 2 Let ε and δ be ultrasmall and let a be observable and not zero. Then

(1) $\varepsilon + \delta \simeq 0$

(2) $\varepsilon \cdot \delta$ is ultrasmall

(3) $\dfrac{a}{\varepsilon}$ is ultralarge

Note the distinction between ultrasmall and ultraclose to zero: if a number is ultraclose to zero, it either can be ultrasmall or zero. For instance, in the first item, if $\varepsilon = -\delta$, we have $\varepsilon + \delta = 0$.

Proof

(1) $\varepsilon + \delta \simeq 0$

$0 \leq |\varepsilon + \delta| \leq 2 \cdot \max\{|\varepsilon|, |\delta|\}$ which is two times an ultrasmall, which is ultrasmall by property 1.

(2) $\varepsilon \cdot \delta$ is ultrasmall

Obvious, but if necessary: $0 < |\delta| < 1$ so $0 < |\varepsilon \cdot \delta| < |\varepsilon|$

(3) If $a \neq 0$ Then: $\dfrac{a}{\varepsilon}$ is ultralarge

Again by contradiction: assume it is not ultralarge, then there is an observable $b > 0$ such that $|\frac{a}{\varepsilon}| = \frac{|a|}{|\varepsilon|} < b \Rightarrow |a| < |b| \cdot |\varepsilon| \simeq 0$, which contradicts that a is observable.

□

The following theorem is very important, since it allows to perform algebraic calculations which involve the concept of ultracloseness.

Theorem 1 (Ultracomputation)
Let a and b be observable with $a \simeq x$ and $b \simeq y$,

(1) $a + b \simeq x + y$

(2) $a - b \simeq x - y$

(3) $a \cdot b \simeq x \cdot y$

(4) If also $b \neq 0$, then $\dfrac{a}{b} \simeq \dfrac{x}{y}$

Proof

(1) – (2) $a \pm b \simeq x \pm y$

Write $x = a + \varepsilon$ and $y = b + \delta$ (which implies that ε and δ are ultrasmall). Then $x + y = a + \varepsilon + b + \delta$ and since $\varepsilon + \delta \simeq 0$ by property 2, page 20 (2) we have the conclusion.

(3) $a \cdot b \simeq x \cdot y$

$x \cdot y = (a + \varepsilon) \cdot (b + \delta) = a \cdot b + a \cdot \delta + b \cdot \varepsilon + \varepsilon \cdot \delta \simeq a \cdot b$ by property 2, page 20 (2) and 2, page 20 (1).

(4) If also $b \neq 0$, $\dfrac{a}{b} \simeq \dfrac{x}{y}$

For this, it is enough to show:

If b is observable and $b \neq 0$ and if $b \simeq y$ then $\dfrac{1}{b} \simeq \dfrac{1}{y}$

and use item 3 to conclude.
Writing $y = b + \delta$ and

$$\frac{1}{y} = \frac{1}{b+\delta} = \frac{1}{b} + h$$

We need to show that $h \simeq 0$
This leads to $b = (1+bh)\underbrace{(b+\delta)}_{\simeq b} \simeq (1+bh) \cdot b$. Therefore we have $b \simeq b \cdot (1+bh)$; divide eacd side by b $1 \simeq 1 + bh$, so $bh \simeq 0$ and since b is observable and not zero, we have $h \simeq 0$

□

Practice exercise PE9 Answer page 123

(1) Give an example of x and y such that $x \simeq y$ but $x^2 \not\simeq y^2$

(2) Give an example of x and y such that $x \simeq y$ but $\frac{1}{x} \not\simeq \frac{1}{y}$

Practice exercise PE10 Answer page 123

Let ε, δ be positive ultrasmall and H, K positive ultralarge numbers.

Determine whether the given expression yields an ultrasmall number, an ultralarge number or a number in between.

(1) $1 + \dfrac{1}{\varepsilon}$

(2) $\dfrac{\sqrt{\delta}}{\delta}$

(3) $\sqrt{H+1} - \sqrt{H-1}$

(4) $\dfrac{H+K}{H \cdot K}$

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7}$

(6) $\dfrac{\sqrt{1+\varepsilon}-2}{\sqrt{1+\delta}}$

Practice exercise PE11 Answer page 124

Find ultrasmall ε and δ (or the relation between them) such that $\dfrac{\varepsilon}{\delta}$ is:

(1) not ultralarge and not ultrasmall,

(2) ultralarge,

(3) ultrasmall.

⚠ The previous exercise shows that if no relation is known between ultrasmall numbers ε and δ, their quotient can be of any possible magnitude.

Symbols

There are no specific symbols for ultrasmall or ultralarge. But N is ultralarge if $\frac{1}{N} \simeq 0$ and ε is ultrasmall if $\varepsilon \simeq 0$ and $\varepsilon \neq 0$.

The only new symbol we introduce is "$\simeq$" which must be distinguished from $\approx$ which is used for (observable) approximations, such as $\pi \approx 3.14$.

Thus $\varepsilon \simeq 0$ reads "epsilon is ultraclose to zero" whereas $\pi \approx 3.14$ read "pi is approximately 3 point one four."

Closure

On page 18, closure is given as:

The result of operations between observable numbers is observable.

It is in fact a consequence of a more general version.

> **Closure (general form)**
> If there is a number satisfying a given property, then there is an observable number satisfying that property.

If we use classical operations which do not refer to observability, such as addition, subtraction, multiplication, divisions, roots, powers, exponentials, etc., then if they have a result, they have an observable result (by closure). Since this result is unique, it is observable; hence, the form given on page 18 is a consequence of the general form.

Let $f : x \mapsto 3x + 2$. If x is always observable, then $f(x)$ is always observable – by closure.

> **Property 3** If a and b are observable and $a \simeq b$, then $a = b$.

Proof

If $a \simeq b$ then $a - b \simeq 0$; which means that $a - b$ is ultrasmall or zero.

By closure (Observability: item 4, page 18), the value is observable, hence cannot be ultrasmall.

□

If $a \simeq b$ then a and b are said to be neighbours. If a is a neighbour of b and is observable then a is an **observable neighbour** of b..

> **Observable neighbour**
> Any number which is not ultralarge has a real number as observable neighbour.

> **Property 4** If x has an observable neighbour, then it is unique.

This method of proof is a classical method in mathematics. To prove that there is only one value having a given property, assume that there are two then prove that they are equal.

Note that it is not always the case. An equation such as $(x-2)(x+3)=0$ has two solutions (2 and -3).

Proof

Assume a and b are observable neighbours of x, then $x = a+\varepsilon$ but also $x = b+\delta$ so $a-b = \delta-\varepsilon \simeq 0$ (by theorem 1 page 21) therefore $a \simeq b$ and by property 3, page 23, $a = b$ $\square$

If a is observable and $\delta \simeq 0$ then $a+\delta \simeq a$ and a is the observable neighbour of $a+\delta$. What the principle above states is that it works both ways: any number x which is not ultralarge can be written in the form $x = a+\delta$ where a is observable and δ is ultrasmall, (or 0 in the case where x is observable).

With the concept of observable neighbour the proof of the fourth item of theorem 1, page 21 can be done in the following way:

Claim:

If b is observable and $b \neq 0$ and if $b \simeq y$ then $\dfrac{1}{b} \simeq \dfrac{1}{y}$

Proof

b is observable and not zero, hence for $y \simeq b$, y is not ultrasmall nor ultralarge. Therefore $\frac{1}{y}$ is not ultralarge nor ultrasmall, hence it has an observable neighbour $c \simeq \frac{1}{y}$ which is not zero. Write $c+\varepsilon = \frac{1}{y} \Rightarrow cy+\varepsilon y = 1$. $\varepsilon y \simeq 0$ by property 2, page 20.

Hence $cy \simeq 1$ and then $\frac{1}{c} \simeq y \simeq b$. But by closure, $\frac{1}{c}$ is observable, so $\frac{1}{c} = b$. So $\frac{1}{y} \simeq \frac{1}{b}$ $\square$

Sets

A great deal of mathematisc is performed by operating on sets of real numbers (per example: a function has an input set and an output set).
Sets can be given by an enumeration: The solution set for the equation $(x-2)(x-3) = 0$ *is* $\{2,3\}$
or by a property, in the form

$$\{x \in \mathbb{R} \mid P(x)\}$$

which reads "all x *which are real numbers and satisfying property* P*" written in mathematical symbols.*
for instance

$$\{n \in \mathbb{N} \mid \exists k \in \mathbb{N}, n = 2k\}$$

is the set of even numbers.

The only new symbol introduced in this course is "$\simeq$*" which refers to the observability of the property and of the variable.*

Properties defining sets thus either do not refer to observability or refer to it by the "$\simeq$*" symbol.*

The same applies to the property mentioned in Closure, page 23.

Properties using ultraclosneess can be used to divide an interval into an ultralarge number of ultrasmall pieces (see page 37, theorem 2)
The set $\{x \in \mathbb{R} \mid x \simeq 0\}$ *may seem to collect all ultrasmall numbers, but* x *determines the observability since it is a statement about* x *so it is observable, and if an observable number is ultraclose to an observable number, they are in fact equal (property 3, page 23). So this set is in simply* $\{0\}$*.*
Note that it is not possible to define a collection of all ultrasmall numbers using a property as defined above. We never need to collect them.

Summary of this chapter

The numbers which can be defined without referring to observability are always observable, or standard.

There are non observable numbers and they have a certain fuzziness: an ultrasmall number cannot be written out using the usual ten digits – or would require using an ultralarge number of zeroes after the decimal point before the first non zero digit.

Closure: *The result of operations between observable numbers is observable.*

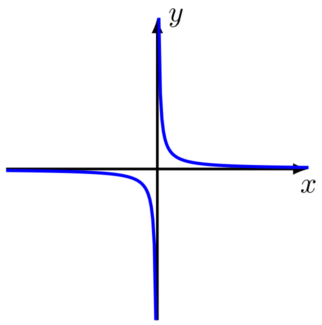
Ultrasmall, ultralarge and ultraclose always refer to an observability.

Chapter 3

Asymptotes

Exercise 3

Consider the real function $f : x \mapsto \frac{1}{x}$



(1) What is the domain of this function?

(2) What happens to the curve close to the vertical axis i.e., for values of very close to 0? Consider x ultraclose to 0.

(3) What happens to the curve close to the horizontal axis? i.e., for very large values of x? Consider ultralarge values of x (positive or negative).

(4) Draw this function for a horizontal domain of $[-100, 100]$ and a vertical range of $[-100, 100]$

Informally: For a given function f, a straight line is **an asymptote** of the function f if it is ultraclose to the graph of the function when either

- x is ultralarge (horizontal or oblique asymptote)
- y (or $f(x)$) is ultralarge (vertical asymptote)

In the following, we will use the symbol a_- to indicate that we choose numbers less than a, and the symbol a_+ to indicate that we choose numbers greater than a.

Definition 5 A real function f **has a vertical asymptote at** $x = a$ if $f(x)$ is positive ultralarge or negative ultralarge for $x \simeq a$.
If it is the case for x greater than a, we write

$$x \simeq a_+ \Rightarrow f(x) \text{ is ultralarge}$$

If it is the case for x less than a, we write

$$x \simeq a_- \Rightarrow f(x) \text{ is ultralarge}$$

If it is the case on both sides of a, we write

$$x \simeq a \Rightarrow f(x) \text{ is ultralarge}$$

Recall that an ultralarge – if nothing is specified – can be either positive or negative.

Definition 6 **(*dx*)** Notation: when an ultrasmall value in the direction of x is used, we will often use dx. It is not the product of two values d and x, it is a single object with a two letter name. It is not zero and can be positive or negative.

Using the notion of dx, "$x \simeq a_+ \Rightarrow f(x)$ is ultralarge" can be rewritten: "for $dx > 0$ $f(a+dx)$ is ultralarge." and "$x \simeq a_- \Rightarrow f(x)$ is ultralarge" can be rewritten:"for $dx < 0$ $f(a + dx)$ is ultralarge." or "for $dx > 0$ $f(a - dx)$ is ultralarge."

Example: The function $f : x \mapsto 1/x$ has a vertical asymptote at 0.
If dx is a positive ultrasmall number then $\frac{1}{dx}$ is positive ultralarge. Hence

$$f(dx) = \frac{1}{dx} \text{ is ultralarge}$$

Exercise 4
Show that $f : x \mapsto \frac{1}{x-2}$ has a vertical asymptote at $x = 2$.
Give the domain of f.

Exercise 5
Show that

$$g : x \mapsto \begin{cases} \dfrac{1}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

has a vertical asymptote at $x = 2$
Give the domain of g.

Exercise 6

Show that $h : x \mapsto \dfrac{|x-2|}{x-2}$ has no vertical asymptote at $x = 2$.
Give the domain of h.

From the previous exercises we can see that there is no immediate link between the fact that values are missing in a domain and the existence of vertical asymptotes.

	values missing in domain	vertical asymptote
$f : x \mapsto \dfrac{1}{x-2}$	yes	yes
$g : x \mapsto \begin{cases} \dfrac{1}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$	no	yes
$h : x \mapsto \dfrac{\|x-2\|}{x-2}$	yes	no

Note that the domain of a function is observable – in the sense that it has same observability as the function. If, on its domain, a function has an extreme value (maximum or minimum) then this extreme value is observable (by Closure – general form, page 23), hence, the function does not take ultralarge values. So if a function takes at least one ultralarge value on its domain then it has no extreme value. In that sense, we say that it reaches infinite values – or extends to infinity [1]*.*

For example the function $: x \mapsto \frac{1}{x-2}$ *takes an ultralarge value for* $x = 2 + dx$ *– whether* dx *is positive or negative. But one can easily see that if we use* $\delta = dx/2$ *then* $f(2+\delta)$ *is larger than* $f(2+dx)$*.*

Hence the closer the value of x is to the value of the vertical asymptote, the larger in absolute value the value of the function will be: there is no extreme value in the neighbourhood of a vertical asymptote.

Definition 7 A real function f has a **horizontal asymptote on the right** (resp. on the left) if there is an observable number L such that

$$x \text{ ultralarge positive (resp. negative) } \Rightarrow f(x) \simeq L$$

Example: Consider

$$\frac{2x^2 - 3x + 1}{x^2 + 1} \text{ for ultralarge } x.$$

This means: consider the fraction for an ultralarge value of x.

[1] see page 6 for a short discussion on the term infinity)

The function $f : x \mapsto \dfrac{2x^2 - 3x + 1}{x^2 + 1}$ is defined on $\mathbb{R}$. $1, 2$ and 3 are always observable. Let x be ultralarge, whether positive or negative. Then

$$f(x) = \frac{2x^2 - 3x + 1}{x^2 + 1} = \frac{x^2(2 - \frac{3}{x} + \frac{1}{x^2})}{x^2(1 + \frac{1}{x^2})} = \frac{2 - \overbrace{\dfrac{3}{x}}^{\simeq 0} + \overbrace{\dfrac{1}{x^2}}^{\simeq 0}}{1 + \underbrace{\dfrac{1}{x^2}}_{\simeq 0}} \simeq \frac{2}{1} = 2$$

hence f has a horizontal asymptote $y = 2$.

Exercise 7

Show that $f : x \mapsto \dfrac{x}{x^2 + 1}$ has a horizontal asymptote at $y = 0$.

Find the value of x for which f crosses its horizontal asymptote.

Exercise 8

Find all asymptotes of

$$f : x \mapsto \begin{cases} \frac{1}{x} & if x \leq 1 \\ x^2 & if x > 1 \end{cases}$$

We now define the oblique asymptote

Definition 8 A real function f has an **oblique asymptote at** $y = ax + b$ on the right (resp. on the left) if there exist observable numbers a, b such that, if x is ultralarge positive (resp. negative), then

$$f(x) - (ax + b) \simeq 0$$

The line $y = ax + b$ is the **oblique asymptote of** f

The existence of an oblique asymptote is a property of f hence the observability is given by f.

This is equivalent to saying that $f(x) \simeq ax + b$ whenever x is ultralarge.

Example: Consider

$$f : x \mapsto \frac{x^3 + 2x^2 + x - 1}{x^2 + 1}$$

Factoring out terms we have

$$f(x) = \frac{x^2(x + 2) + x + 2 - 3}{x^2 + 1} = \frac{(x + 2)(x^2 + 1) - 3}{x^2 + 1}$$

Therefore

$$f(x) = x + 2 - \frac{3}{x^2 + 1}$$

Let x be ultralarge. We have

$$f(x) - (x+2) = \frac{-3}{x^2+1} \simeq 0$$

because x^2+1 is ultralarge. Hence f has an oblique asymptote at $y = x+2$, i.e., $a = 1$ and $b = 2$

Exercise 9

Find the asymptotes (if any) of

(1) $f_1 : x \mapsto \dfrac{x}{2x^2+1}$

(2) $f_2 : x \mapsto \dfrac{2x^2+1}{x}$

(3) $f_3 : x \mapsto \dfrac{x^3+2}{2x^2-1}$

(4) $f_4 : x \mapsto \dfrac{x^2+2x+1}{x+1}$

(5) $f_5 : x \mapsto \dfrac{3x^3+2x^2-x+12}{x^2+8}$

For functions which are not rational functions, where the polynomial long division does not apply, we have the following:

Property 5 Let f be a real function and let a and b be observable (relative to f). Then f has an oblique asymptote at $y = ax+b$ on the right (resp. on the left) if and only if there are observable a and b such that for ultralarge positive (resp. negative) x,

$$\frac{f(x)}{x} \simeq a \quad \text{and} \quad (f(x) - ax) \simeq b$$

Remark: If $a = 0$ the line $y = ax+b$ becomes $y = b$ i.e., a horizontal asymptote.

Proof

Since the asymptote is a property of the function, the observability is given by f but not by x.

If f has an oblique asymptote $y = ax+b$ then for ultralarge x, we have $f(x) \simeq ax+b$. Divide by x:

$$\frac{f(x)}{x} \simeq a + \underbrace{\frac{b}{x}}_{\simeq 0} \simeq a$$

.

Furthermore, if we have, for ultralarge x, $f(x) \simeq ax+b$, then $f(x) - ax \simeq b$

Conversely, assume that for ultralarge x, $\frac{f(x)}{x} \simeq a$ and $f(x) - ax \simeq b$, then it is immediate that for ultralarge x, $f(x) \simeq ax+b$ □

Example: Consider $f : x \mapsto \sqrt{x^2+1}$ defined on $\mathbb{R}$. Let x be positive ultralarge. Then

$$\frac{f(x)}{x} = \frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2(1+1/x^2)}}{x} = \frac{|x|\overbrace{\sqrt{1+1/x^2}}^{\simeq 1}}{x} \simeq \begin{cases} 1 & \text{it } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Moreover:

$$f(x) - x = \sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1} - x)\cdot(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \frac{1}{\sqrt{x^2+1} + x} \simeq 0$$

Hence f has an oblique asymptote at $y = x$ on the right.

On the left, the function has an oblique asymptote at $y = -x$

Exercise 10

Find the asymptotes (if any) of

$$f : x \mapsto x^{\frac{3}{2}}$$

Practice exercise PE12 Answer page 124

Find all asymptotes of the following functions.

(1) $f_1 : x \mapsto \dfrac{x^2 - x}{x - 1}$

(2) $f_2 : x \mapsto \dfrac{4x^3 + 2x^2 - 5}{3x^3 - 4x^2}$

(3) $f_3 : x \mapsto \sqrt{x^2 + x}$

(4) $f_4 : x \mapsto \dfrac{\sqrt{x^5 + x}}{\sqrt{3x^5 - x}}$

(5) $f_7 : x \mapsto \dfrac{x^{10}}{x^{10} + 1}$

Summary of this chapter

Ultraclose to a specific value of the independent variable x, some curves are indistinguishable from a vertical line: this is a vertical asymptote.

It is also possible that some curves are indistinguishable from a horizontal line when x is ultralarge. This horizontal line is called a horizontal asymptote.

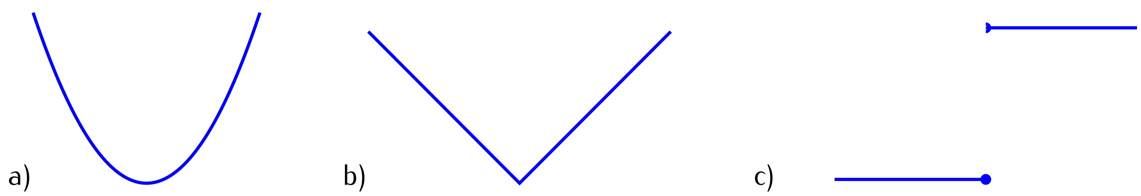
Oblique asymptotes are similarly defined (for ultralarge values of x).

The equations of these asymptotes are determined by methods which involve using ultrasmall or ultralarge numbers in computations.

Chapter 4

Steps

In practice exercises 6, 7 and 8, page 14, we have seen functions whose graphs look like the following lines.



Line a) is smooth.
Line b) has a sharp bend at one value of x; it is pointed.
For line(s) c) there is a step.

One way to picture these situations is to imagine a flea walking on the line on a foggy day. Situation a) provides for an easy trip. The flea can walk by dragging its legs on the line. Situation b) is a bit uncomfortable at the pointed value but it is still possible for the flea to advance carefully by dragging its legs. Maybe the flea will stumble a bit, but will not fall off the line. Situation c) is totally different: the flea must jump, but it cannot see how far to jump! If it drags its legs it might fall off. And these situations do not depend on which direction the flea is travelling along the line.

Curves a) and b) are **continuous**
Curve c) is **not continuous**.

Informally:
A function is continuous at a given value of $x = a$ if you can draw its graph around that value without lifting the pencil.
Rephrasing the sentence, we can also say that a function is continuous at a given value of $x = a$ if it is where you would expect it to be by observing where it is just before and just after that value.

This last sentence is formalised in the following way:

Definition 9 (Continuity) Let f be a real function defined around a[a]. We say that f **is continuous at** a if

$$x \simeq a \Rightarrow f(x) \simeq f(a)$$

[a] This means that f is defined on an interval extending on either side of a

This definition refers to an ultraproximity. The observability (see page 18) of this situation is given by a and the parameters used in defining the function f. This simply means that we adapt our telescopes/microscopes in such a manner that all these values are observable. By closure (see page 18) the value $f(a)$ is also observable.

Note that "$x \simeq a$" does not depend on a specific value x as long as it satisfies the condition of being ultraclose to a. It can be on the right hand side or on the left hand side of a.

x can be written $a + dx$ (for some ultrasmall dx), which leads to:

Let f be a real function defined around a. We say that f **is continuous at** a if

$$f(a + dx) \simeq f(a) \qquad \text{for any } dx$$

Here, dx must not be specific (provided it is ultrasmall) and can be positive or negative.

Exercise 11

Show that $f : x \mapsto x^3$ is continuous at $a = 2$

Exercise 12

Show whether $f : x \mapsto \dfrac{x}{x^2+1}$ is continuous for all values of x.

Exercise 13

(1) Show that $f : x \mapsto |x|$ is continuous at $x = 0$, at $x = 1$, at $x = -1$ and at x in general.

(2) Show that $g : x \mapsto \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$ and at x in general.

(3) Show that $h : x \mapsto \begin{cases} x^2 & \text{if } x \geq -1 \\ x^3 & \text{if } x < -1 \end{cases}$ is not continuous at $x = -1$ but is continuous for all other values of x.

Definition 10 (One-sided continuity) A function f is continuous on the right (resp. on the left) of a if $f(x) \simeq f(a)$ whenever $x \simeq a_+$ (resp. $x \simeq a_-$).

Definition 11 (Continuity on an interval)

(1) Let f be a real function defined on the open interval $]a, b[$. Then f is **continuous on** $]a, b[$ if it is continuous at all $x \in]a, b[$.

(2) Let f be a real function defined on the closed interval $[a, b]$. Then f is **continuous on** $[a, b]$ if it is continuous at all $x \in]a, b[$ and continuous on the right at a and on the left at b.

Exercise 14

Show, using the definition of continuity, whether the following functions are continuous on the given intervals.

(1) $f : x \mapsto \frac{1}{3}x + \sqrt{2}$ on $\mathbb{R}$

(2) $g : x \mapsto x^2 - 3x - 1$ on $\mathbb{R}$

(3) $h : x \mapsto \dfrac{x+2}{x-1}$ on $]1, +\infty[$

Exercise 15

Determine whether $f : x \mapsto x^2$ is continuous on its domain.

Exercise 16

Determine whether $f : x \mapsto \frac{1}{x}$ is continuous on its domain.

Exercise 17

Prove that $x \mapsto \sqrt{x}$ is continuous on its domain.

Property 6 Let f and g be two real functions continuous at a. Then

(1) $f \pm g$ is continuous at a.

(2) $f \cdot g$ is continuous at a.

(3) $\dfrac{f}{g}$ is continuous at a if $g(a) \neq 0$.

Proof

For brevity, let's call $f(a) = b$ and $g(a) = c$. Let's now introduce two dependent variables, $f(x) = u$ and $g(x) = v$.
By continuity, for $x \simeq a$ we have $u \simeq b$ and $v \simeq c$. Now, by theorem 1, page 21, we have $b \pm c \simeq u \pm v$ $\quad b \cdot c \simeq u \cdot v$ and $\frac{b}{c} \simeq \frac{u}{v}$, which exactly express the definition of continuity at a for, respectively, $f \pm g$, $f \cdot g$ and $\dfrac{f}{g}$. $\square$

Clearly, if f and g are continuous on an interval I then the sum, difference, product and quotient (if $g(x) \neq 0$ for all $x \in I$) are continuous on I.

When considering two functions, they could have different observability. Say f contains a parameter h which is less observable than g. Continuity of f is observed under its own observability and similarly for g; but for the continuity of $f+g$ the global observability of f and g must be used: the observability of g must be extended to contain h. Yet notice that when we consider continuity of g at a, if $g(a+dx) \simeq g(a)$ all we require is that dx be ultrasmall; it does not matter if is even smaller.

Properties do not change when the observability is extended.

Property 7 Let f and g be two real functions. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Reminder: $g \circ f$ is the composition of f first, then g applied to the result of f. $(g \circ f)(x)$ is also written $g(f(x))$.

Proof

$x \simeq a \Rightarrow f(x) \simeq f(a)$ by continuity of f at a and $g(f(x)) \simeq g(f(a))$ by continuity of g at $f(a)$.

□

Similarly to what stated for property 6, page 35, if g is continuous on an interval containing $f(I)$ then $g \circ f$ is continuous on I.

Exercise 18

The converse of property 6, page 35, does not necessarily hold. Find functions f and g (either by a rule or graphically) such that

(1) $f+g$ is continuous at 1 but at least one of f or g is not continuous at 1

(2) $f \cdot g$ is continuous at 1 but at least one of f or g is not continuous at 1

(3) $\dfrac{f}{g}$ is continuous at 1 but at least one of f or g is not continuous at 1 (with $g(1) \neq 0$)

Two difficult theorems

The two following theorems may seem to state something obvious when in fact they state a rather subtle property of real numbers. To see this we leave for a moment the study of real functions and look at a function from rational numbers to rational numbers.

Consider

$$\begin{array}{rcl} \mathbb{Q} & \to & \mathbb{Q} \\ f : x & \mapsto & x^2 - 2 \end{array}$$

There are values such that $f(x) < 0$ (for instance, $f(0) = -2$) and values such that $f(x) > 0$ (for instance, $f(2) = 2$), yet since the square root of two is not a rational number, there is no value such that $f(x) = 0$. Nonetheless, the definition of continuity makes sense in $\mathbb{Q}$ so the following theorem is in fact a characterisation of real numbers as much as it is a theorem about continuous functions.

Take an ultralarge number of digits of $\sqrt{2}$. Because this decimal expansion is finite (though it contains an ultralarge number of digits) it is a rational number q (hence in $\mathbb{Q}$) whose square is ultraclose to 2 by continuity of the square root function (see exercise 17). If we were in $\mathbb{R}$ we would say it is ultraclose to $\sqrt{2}$ which is observable – by closure).

If a rational number is always observable, then it remains always observable when considered as a real number. So if q has a rational observable neighbour, it would also have $\sqrt{2}$ as observable neighbour in $\mathbb{R}$ – which means that q would have two different observable neighbours in $\mathbb{R}$ which has been shown to be impossible (see page 23). So the existence of an observable neighbour is a property of real numbers. It is in fact one of their main characterisations.

What the following theorem states is in fact that real numbers are sufficiently complete to provide a solution to any equation satisfying the conditions of the theorem.

Theorem 2 (Intermediate value)
Let f be a real function continuous on $[a, b]$ with $f(a) \neq f(b)$. Let d be a real number between $f(a)$ and $f(b)$. Then there exists c in $[a, b]$ such that $f(c) = d$.

This theorem does not tell us how to find the root or the value c such that $f(c) = d$. It only asserts the existence of such a number. For specific functions where we can calculate explicitly the roots this theorem is not really necessary but, when proving theorems about continuous functions in general, it is the only way to know that there is a root.

Proof

Observability is determined by f, a, b and d.

We first assume that $f(a) < f(b)$.

Let N be an ultralarge integer, and $dx = \frac{b-a}{N}$ (therefore $dx \simeq 0$) and let $x_k = a + k \cdot dx$, for k between 0 and N. We thus have $x_0 = a,\ x_1 = x_0 + dx,\ \ldots,\ x_N = b$.

Let x_j be the first element of the partition $\{a, x_1, x_2, \ldots, x_N = b\}$ such that $f(x_j) \leq d$ and $f(x_{j+1}) \geq d$

Since $a \leq x_j \leq b$, this x_j it is not ultralarge, and therefore has an observable neighbour c, so $x_j \simeq c$ and $x_{j+1} \simeq c$. By closure $f(c)$ is observable with $f(c) \simeq f(x_j) < d$ and $f(c) \simeq f(x_{j+1}) \geq d$, hence $f(c) = d$ (since d is observable).

If $f(a) > f(b)$ the same proof holds with inequalities reversed. Or alternatively use $g(x) = -f(x)$.

□

Comment: *Something fundamental and new is used in this proof. The real numbers and the rational numbers are dense sets (see page 7) so it is not possible to cover an interval of such numbers by passing from one value to the next (until a solution is found for example) since there is no next value. This is very different from finite sets: these can be numbered with natural numbers, which have a very important property: every nonempty set of natural numbers has a smallest element (even if we do not know what it is). Hence by dividing the interval $[a, b]$ into* *an ultralarge number of values, this new set can be divided into A, the x_k that satisfy $f(x_k) \leq d$ and B, those that satisfy $f(x_k) > d$. Note that A and B can have no element in common. (The function my have waves and go above and below the value d several times, we only need to find one answer to prove the theorem.) B has a smallest element which may not necessarily be the exact solution but the proof shows that its observable neighbour is the required solution.*

This was the first use of a method frequently used in analysis: using ultrasmall values we find an approximation to the solution of the problem and check whether the observable neighbour is the answer.

Exercise 19

Give an example of a function f discontinuous on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$ such that there is no c in the interval $[a, b]$ such that $f(c) = 0$.

Give an example of another function g discontinuous on $[a, b]$ with $g(a) < 0$ and $g(b) > 0$ such that there is a c in the interval $[a, b]$ such that $g(c) = 0$.

Definition 12 A function has a **maximum value** (respectively **minimum value**) on an interval I if there is a $c \in I$ such that for any $x \in I$ we have $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$).
If a point is either a maximum or a minimum, it is an **extremum**.

We say that f has a maximum at c. The maximum *point* is $\langle c, f(c) \rangle$

Note that

$$\begin{array}{rcl} [0,1[& \rightarrow & \mathbb{R} \\ f : x & \mapsto & x+1 \end{array}$$

has no maximum. In fact, for every real number $c \in [0, 1[$ that we can conceive, no matter how ultraclose to 1, there will always be a real number d such that $c < d < 1$ and, consequently, $f(d) > f(c)$. This is due to the fact that we define the function over an interval which is open on the right. On the other hand, function f has a minimum at $x = 0$, since for every $x \in [0, 1[$ we have $f(x) \geq f(0) = 1$

For the proof of the following theorem, the general form of closure is used.

Theorem 3 (Extreme value)
Let f be a function continuous on $[a, b]$
Then f has a maximum and a minimum on $[a, b]$

The proof of this theorem requires a step which is simpler to prove separately and refer to in the proof.

Property 8 If $x \in [a, b]$ and c is observable with $c \simeq x$, then $c \in [a, b]$

Proof

Assume that $c \notin [a, b]$ and $c > b$. (The same proof holds for $c < a$ by reversing inequalities).

Then we would have $x \leq b < c$ with $x \simeq c$, which implies that $x \simeq b$ and that $b \simeq c$. But by property 4 (page 23), we have $b = c$. Hence c is not outside the interval.

□

Proof

(Of theorem)

Closure in the contrapositive: a statement and its negation have same parameters. If a statement is true for all observable values of a set, then it is true for all values in that set. If it were not, there would be a counterexample, but by closure, if there is a counterexample, there is an observable one. So there is no counterexample.

Take an ultralarge positive integer N and partition $[a, b]$ into N even intervals, each of length $dx = \frac{b-a}{N}$. We thus have $x_0 = a,\ x_1 = x_0 + dx,\ \ldots,\ x_N = b$.

Call x_M the first point of the partition such that $f(x_M) \geq f(x_i)$ for any i between 0 and N (the maximum on the partition).

Let c be the observable neighbour of x_M. Because the interval is closed, c is in the interval, by property 8. By closure $f(c)$ is observable.

Let $x \in [a, b]$ be observable. By closure $f(x)$ is observable.

Claim: $f(x) \leq f(c)$.

Proof of this claim: since $x \in [a, b]$, there is an i such that $x_i \leq x \leq x_{i+1}$. Then $f(x) \simeq f(x_i) \leq f(x_M) \simeq f(c)$. So either $f(x_i) < f(x_M)$ and the conclusion is that $f(x) < f(c)$, or $f(x_i) = f(x_M)$ and the conclusion is that $f(x) = c$ by uniqueness of the observable neighbour (see page 23). In both cases, $f(x)$ is not above $f(c)c$.

This shows that $f(c)$ is the maximum over all observable numbers. By closure in the contrapositive form, it is the maximum over all real numbers in the interval. □

Exercise 20

The extreme value theorem has the hypothesis that the function is continuous on a closed interval. These hypotheses are necessary in the proof but do not imply that a function cannot have extreme points in other situations or that a function not satisfying the hypotheses cannot have extreme points.

Find examples of different functions such that

(1) f: its domain is an open interval and f has no maximum (or no minimum)

(2) g: its domain is a closed interval but g is not continuous and has no maximum.

(3) h: its domain is an open interval, h is not continuous, yet it has a maximum.

Summary of this chapter

Most functions you will encounter are continuous except maybe at a few points of their domain.

Note that $f : x \mapsto \frac{1}{x}$ is continuous (on its whole domain). The only point where it could seem to be discontinuous is at $x = 0$, which is not in the domain, hence not part of the function. For a function to be discontinuous, it must be possible to specify a point a such that there is another point x with $x \simeq a$ such that $f(x) \not\simeq f(a)$. If $f(a)$ does not exist, the proof of discontinuity is void, hence f is continuous.

Properties 6, page 35 and 7, page 36 show that continuity can be checked by analysing the given function as a combination of simpler functions.

***The intermediate value theorem** (theorem 2, page 37) is a characterisation of the real numbers and real functions. Its proof relies on the existence, in the real numbers, of the observable neighbour (see page 23).*

***The extreme value theorem** (theorem 3, page 38) makes a link between closed intervals and continuity to prove the existence of maximum points and minimum points. It does not imply that without these conditions a function cannot have an extremum.*

Chapter 5

Slopes

As before (see page 28), we use dx to indicate an ultrasmall *increment* of the variable x. It may be positive or negative but will never be chosen to be 0.

Exercise 21
Let

$$f : x \mapsto x^2$$

The graph of this function is a curve (a parabola). Zoom in on the point $\langle 3, 9 \rangle$. 3 and 9 are always observable. Consider the value of the function at $3 + dx$, (for dx ultrasmall as mentioned above) and draw a straight line passing through $\langle 3, 9 \rangle$ and $\langle 3 + dx, f(3 + dx) \rangle$.

- What is the slope of this straight line?
- What observable value is this slope ultraclose to?

Definition 13 (Derivative) Let f be a real function defined on an interval I with $a \in I$.
If there is an observable value D such that

$$\frac{f(a + dx) - f(a)}{dx} \simeq D$$

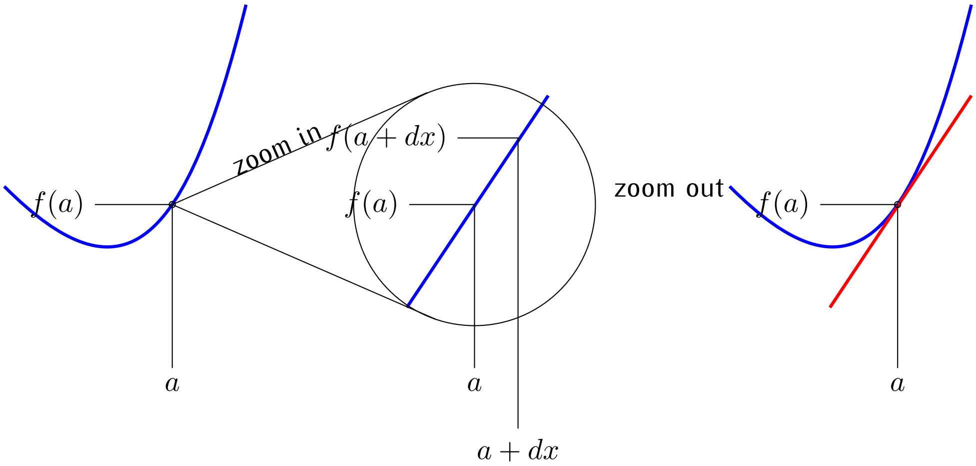
not depending on ultrasmall dx then D is the **derivative** of f at a.

Notation: the derivative of f at a is noted $f'(a)$.

When $f'(a)$ exists,

- we say that f is **differentiable at** a
- it is the **slope** of f at $x = a$.
- it is the **observable neighbour** of $\dfrac{f(a + dx) - f(a)}{dx}$

Metaphorically, finding the derivative can be described by: Zoom in. If what you see is indiscernible from a straight line, then measure the slope of that line. Extend that line and zoom out.



Exercise 22

Using definition 13, page 41 calculate the derivative of: $f : x \mapsto 3x^2 + x - 5$ at $x = -2$ and $x = 2$.

Exercise 23

Using definition 13 calculate the derivatives (if they exist) of the following:

(1) $g : x \mapsto 2x^3 - 2$ at $x = 1$ and $x = 0$.

(2) $h : x \mapsto |x|$ at $x = 2$, $x = -2$ and at $x = 0$.

Exercise 24

Let $f : x \mapsto x^3 - 3x - 2$. Check that 2 is a root of f. Are there other roots?

At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x do we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

Exercise 25

Let $f : x \mapsto 2x^3 - 4x^2 + 2x$. At what values of x is the function equal to zero? At what values of x is the derivative equal to zero? What is the value of the function at these points? At what values of x de we have $f'(x) > 0$ and at what values do we have $f'(x) < 0$?

Use all this information to make a rough sketch of the function.

Practice exercise PE13 Answer page 124

Calculate the derivative of the following:

(1) $f : x \mapsto 5x^2 - 10x$ at $x = 2$

(2) $g : x \mapsto 5(x - 10)^2$ at $x = 3$

(3) $h : x \mapsto x^4 + x^3 + x^2 + x + 1$ at $x = 1$

(4) $k : x \mapsto 5x^2 + 10$ at $x = 2$

Practice exercise PE14 Answer page 124

Consider the derivative at x (general case) of the function

$$f : x \mapsto x^2 + 3x$$

Show that it is differentiable for all x and that $f'(x) = 2x + 3$.

Note that in the computation of a derivative, the division is **always** between two ultrasmall numbers. They <u>cannot</u> be replaced by 0 since $\frac{0}{0}$ is not defined and ultrasmall numbers are, by definition, different from zero.

Definition 14 If a function is differentiable for all x on a given interval I, (for any $x \in I$ the value $f'(x)$ exists) then **the derivative function** is

$$f' : x \mapsto f'(x)$$

Practice exercise PE15 Answer page 125

Using definition 13, give the derivative functions of $f : x \mapsto x^2$ and $g : x \mapsto x^3$

Exercise 26

Using definition 13, give the derivative functions of the following functions:

(1) $f : x \mapsto 3x + 2$

(2) $g : x \mapsto 2x^2 - x$

(3) $h : x \mapsto 5x^3 + 2x^2 - x$

(4) $k : x \mapsto 5x^3 + 2x^2 + 3x + 2$

The Increment

Notation: Let dx be ultrasmall relative to f and x. We define **the increment of** f **at** a as:

$$\Delta f(a) = f(a + dx) - f(a) \text{ or } f(a + dx) = f(a) + \Delta f(a)$$

Comment: *$\Delta f(a)$ is the variation of a function corresponding to a given variation (or increment) of the independent variable and is therefore in fact something depending on two variables: a and the corresponding variation of x, which can be written Δx. If g is a growth function, the independent variable would be time, possibly noted as t and could represent years and Δt could be one year. In analysis, the increment of the independent variable will always be taken ultrasmall (and usually written dx).*

Formally one could write $\Delta f(a, \Delta x)$. The growth function would be $\Delta g(t, year)$. In practice this proves to be cumbersome and the reference to the increment is omitted. In physics Δx often refers to a finite variation whereas in analysis it refers to an ultrasmall variation of the independent variable. The magnitude of Δf is to be determined and could be anything from ultrasmall to ultralarge.

Note that if $f : x \mapsto \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

then $\Delta f(0) = \begin{cases} -1 & \text{if } dx < 0 \\ 1 & \text{if } dx > 0 \end{cases}$ *because f is not continuous at 0.*

From definition 13, upon dividing by dx, we have:

$$\frac{\Delta f(a)}{dx} \simeq f'(a)$$

Notation: A "$\simeq$" symbol may be replaced by a "$=$" symbol by adding a value ultraclose to zero on one of the sides i.e., $a \simeq b \Rightarrow a = b + \varepsilon$ where $\varepsilon \simeq 0$.

Hence

$$\frac{\Delta f(a)}{dx} = f'(a) + \varepsilon \text{ with } \varepsilon \simeq 0$$

and

Property 9 (Increment equation) Let f by differentiable at a, then

$$\Delta f(a) = f'(a) \cdot dx + \varepsilon \cdot dx$$

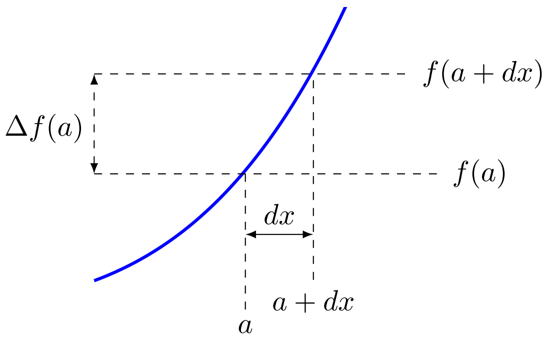
or equivalently

$$f(a + dx) = f(a) + f'(a) \cdot dx + \varepsilon \cdot dx$$

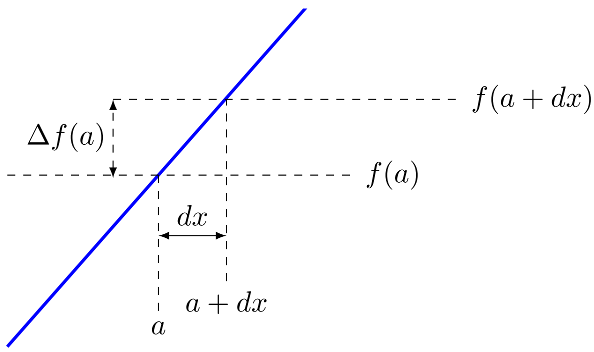
$f'(a)$ is **NOT equal** to $\dfrac{\Delta f(a)}{dx}$.

$$f'(a) \simeq \frac{\Delta f(a)}{dx}$$

The relation is one of ultraproximity.



When we zoom on smooth functions, we will notice that they tend to look almost like straight lines, hence the previous drawing would be more like the following:



Note: drawings involving ultrasmall or ultralarge values are not meant to be to scale nor be a correct representation. Their purpose is merely to help the mind.

Continuity revisited The definition of continuity (page 34) can be rewritten

f is continuous at a if $\Delta f(a) \simeq 0$

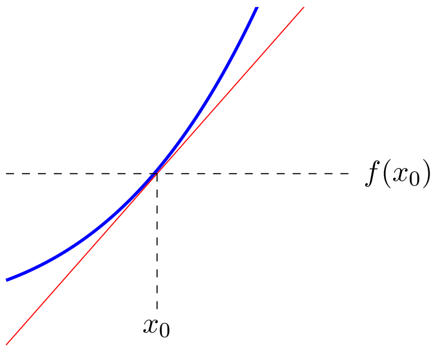
Property 10 If a real function f is differentiable at a then f is continuous at a.

Proof
By the increment equation.
$\Delta f(a) = \underbrace{f'(a) \cdot dx}_{\text{observable} \times \text{ultrasmall } \simeq 0} + \underbrace{\varepsilon \cdot dx}_{\simeq 0}$ □

Exercise 27
Note that the converse does not necessarily hold. Find a function which is continuous at $x_0 = 0$ and not differentiable at that point.

Tangent line

Suppose f is differentiable at x_0. We observe that through a microscope, the curve of the function at x_0 is indistinguishable from a straight line passing through $\langle x_0, f(x_0) \rangle$ whose slope is $f'(x_0)$. This line is the tangent line.



Definition 15 Let f be differentiable at x_0. The tangent line T is the straight line that satisfies $T(x_0) = f(x_0)$ and $T'(x_0) = f'(x_0)$.

Exercise 28
Let $f : x \mapsto x^2$. Find the equation of the straight line tangent to f at $x_0 = 3$.

Exercise 29
Show that

$$T_{x_0} : x \mapsto f'(x_0)(x - x_0) + f(x_0)$$

Exercise 30

Give the equation of the line tangent to $x \mapsto x^3 - 3 \cdot x^2$ at $x_0 = 2$.
For which values of x is this tangent horizontal?

The differential

Definition 16 Let f be a real function differentiable on an interval around a. Let dx be ultrasmall. The **differential of** f **at** a, written $df(a)$, is

$$df(a) = f'(a) \cdot dx$$

Thus

$$\frac{df(a)}{dx} = f'(a)$$

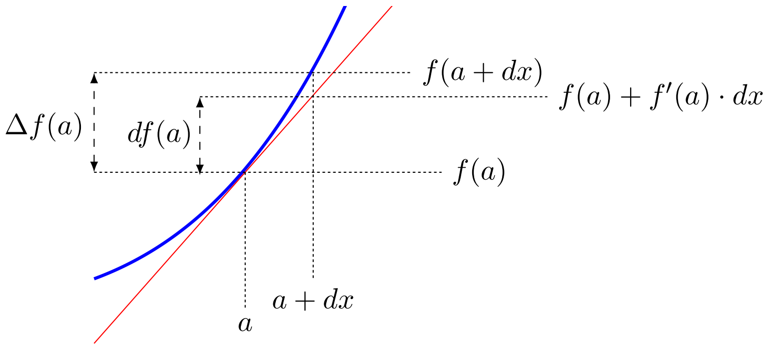
or still (if we use $y = f(a)$)

$$\frac{dy}{dx} = y'$$

If f is differentiable the following holds:

$$\frac{\Delta f(a)}{dx} \simeq \frac{df(a)}{dx} = f'(a)$$

Whereas $\Delta f(a)$ is the variation of the function, the differential is the variation along the tangent line.



By the definition of increment we have: $f(x+dx) = f(x) + \Delta f(x)$ and, with the increment equation (page 44), the following relations hold:

$$\begin{aligned} f(x+dx) &= f(x) + \Delta f(x) \\ &= f(x) + f'(x) \cdot dx + \varepsilon \cdot dx \qquad \text{for } \varepsilon \simeq 0 \\ &= f(x) + df(x) + \varepsilon \cdot dx \end{aligned}$$

Note that while dx is ultrasmall, Δy and dy are ultraclose to zero.

Critical points

Differentiability is defined on an open interval, since the derivative is defined only if the observable part of the fraction $\frac{\Delta f(x)}{dx}$ is the same for $dx > 0$ and $dx < 0$.

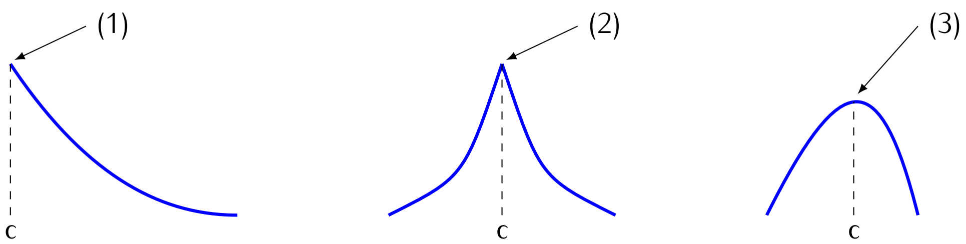
To ensure that nothing "bad" happens at the endpoints of an interval, the following theorems and properties specify that f is differentiable on an open interval $]a, b[$ and continuous on the closed interval $[a, b]$.

Theorem 4 (Critical Point Theorem)
Let f be a continuous function on an interval I and suppose that c is a point in I such that f has either a maximum or a minimum at c.
Then one of the following three things must happen:

(1) c is an endpoint of I.

(2) $f'(c)$ is undefined.

(3) $f'(c) = 0$



Proof

The first two situations do not refer to the derivative except to indicate (in the second case) that there is none. For the third case:

Assume that $\langle c, f(c) \rangle$ is a local maximum. (The same proof holds for a minimum.) Then $f(c) \geq f(c + dx) \Rightarrow f(c + dx) - f(c) \leq 0$.

Let dx be positive, then $f'(c) \simeq \frac{f(c+dx)-f(c)}{dx} \leq 0$ since, by hypothesis, $f'(c)$ exists.

Let dx be negative, then $f'(c) \simeq \frac{f(c+dx)-f(c)}{dx} \geq 0$.

The only observable number which is ultraclose to positive and negative values is 0. $\square$

Exercise 31

Find the derivative of $f : x \mapsto x^3$ at $x = 0$ to see that the converse of theorem 4 does not hold.

⚠ Exercises which require to prove that the converse of a theorem does not necessarily hold are extremely important! A theorem has hypotheses (or conditions) and a conclusion. If the theorem is in the form "if and only if" then satisfying the hypotheses or satisfying the conclusion are equivalent (the Pythagorean theorem is an example). But in analysis, most theorems are in the form "if the hypotheses are satisfied then the conclusion is true". They say nothing for the situation where the hypotheses are not satisfied, nor is it possible to deduce that the hypotheses are true if the conclusion seems correct. Consider the hypothesis "It is raining" and the conclusion "the roads are wet" – and we accept as true the statement "If is is raining, then the roads are wet", then if is not raining, we cannot deduce that the roads are not wet. Maybe somebody is playing with a water hose. For the same reason, we cannot deduce anything from the fact that the roads are wet. But the contrapositive is true: if the roads are not wet, then it is not raining.

Theorem 5 (Rolle)
Let f be a real function continuous on $[a, b]$ and differentiable on $]a, b[$. If $f(a) = f(b)$, then there is a $c \in]a, b[$ such that

$$f'(c) = 0$$

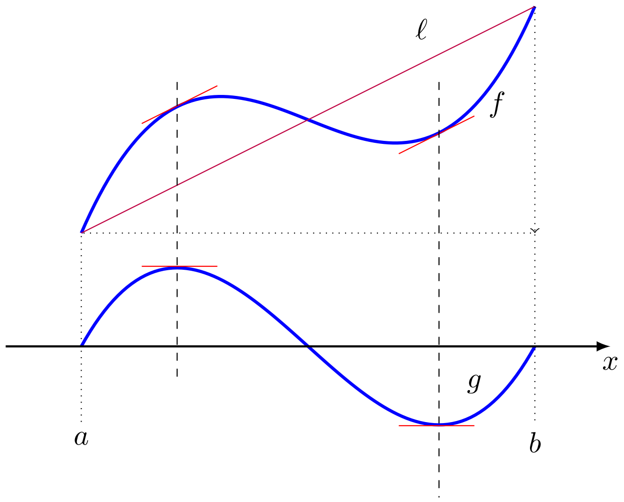
Exercise 32

Prove Rolle's theorem.

Theorem 6 (Mean value)
Let f be a real function continuous on $[a, b]$ and differentiable on $]a, b[$. Then there is a $c \in]a, b[$ such that

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

The starting point of the proof is the following: consider g which is obtained by subtracting the line $\ell(x)$ through $(a, f(a))$ and $(b, f(b))$ from the function f i.e., $g(x) = f(x) - \ell(x)$.



Exercise 33

Show that g satisfies Rolle's theorem and conclude the proof of the mean value theorem.

> **Property 11** Let f and g be functions and I an interval.
>
> $$f' = g' \iff \text{there is a real number } C \text{ such that } f = g + C$$

Exercise 34

Prove the property 11.

One direction of property 11 comes from the fact that $c' = 0$. You will need theorem 6, page 49) for the other direction.

Variation

> **Definition 17** Let f be a real function defined on an interval I.
>
> (1) The function f is **increasing on** I if $f(x) \leq f(y)$, whenever $x < y$ in I.
>
> (2) The function f is **decreasing on** I if $f(x) \geq f(y)$, whenever $x < y$ in I.

If the inequalities on f are strict (meaning: the equality cases are excluded), then we say that the function is strictly increasing or strictly decreasing.

Property 12 (Variation and derivative) Let f be a real function differentiable on an interval $]a, b[$ and continuous on $[a, b]$. Then

(1) If $f'(x) \geq 0$ (> 0) whenever $x \in]a, b[$ then f is (resp. strictly) increasing on $[a, b]$.

(2) If $f'(x) \leq 0$ (< 0) whenever $x \in]a, b[$ then f is (resp. strictly) decreasing on $[a, b]$.

(3) If $f'(x) = 0$ whenever $x \in]a, b[$ then f is constant on $[a, b]$.

The converse is obvious: if f is increasing at a, then $f'(a) \geq 0$, etc.

Proof

Assume $f'(x) \geq 0$ (> 0) whenever $x \in]a, b[$. By the mean value theorem (theorem 6, page 49) for any x and y in$]a, b[$ with $x < y$, there is a $c \in]x, y[$ such that $f(y) - f(x) = f'(c)(y - x)$. Since $y - x > 0$ and $f'(c) \geq 0$ we have $f(y) \geq f(x)$ hence f is increasing on the interval.

If $f'(x) \leq 0$ (< 0) or $f'(x) = 0$ the same argument proves that the function is decreasing or constant. □

Summary of this chapter

The derivative gives a value for the slope of a function at a given point and leads to the equation of a straight line tangent to a curve.

If a differentiable function defined on an interval has a maximum or a minimum at a given interior point, then its derivative is zero at that point. The converse is not true: a derivative can be zero without that point being an extreme value. Moreover, if the function has a maximum or a minimum at an endpoint of its interval, the derivative of the function at that point is not necessarily zero.

The mean value theorem (theorem 6, page 49) has important consequences. In particular the fact that if $f'(x) = 0$ on a whole interval, the function is constant (property 12, page 51). This is not as obvious as it could seem. The derivative is zero if $\frac{\Delta f(x)}{dx} \simeq 0$: note that it is not an equal sign. The fact that the derivative is zero everywhere on the interval is a consequence of the mean value theorem.

If two functions have the same derivative (property 11), they are the same functions up to an additive constant. (If the function is the position of an object, the derivative represents its velocity. If two objects move with same velocity, the distance between them remains constant.)

All this information about the derivative is put in property 12 and leads to the possibility of visualising the curve: when the derivative is positive the function is going upwards (from left to right), when it is negative, it goes down, and if it is zero, the curve is horizontal (possibly just at one point).

The differential $df(x) = f'(x) \cdot dx$ or $dy = y' \cdot dx$ is an ultrasmall value: it represents the variation along the tangent line.

Chapter 6

Differentiation rules

We have seen so far that definition 13 (41) allows us to calculate the derivatives of functions; however, to calculate the derivatives of complex functions, we might benefit from more powerful tools. Often, evaluating the derivatives of complex functions is better done by breaking down these functions into simpler ones, of which we can easier calculate the derivatives, and then compose the results.

Since observable numbers remain observable if we zoom further in, a property is not changed if the observability is extended. So when considering combinations of functions, the observability is extended if necessary to contain all parameters of both functions (see page 36).

According to definition 13, for every proof of the properties in this chapter, we will evaluate the increment of the complete function divided by the ultrasmall increment dx, and we will check what observable value this is ultraclose to.

Product of functions

Using **dependent variables** it is usual to write $f(x) = y$ but when two functions are involved, we use $f(x) = u$ and $g(x) = v$, (then $\Delta f(x) = \Delta u$ and $\Delta g(x) = \Delta v$).

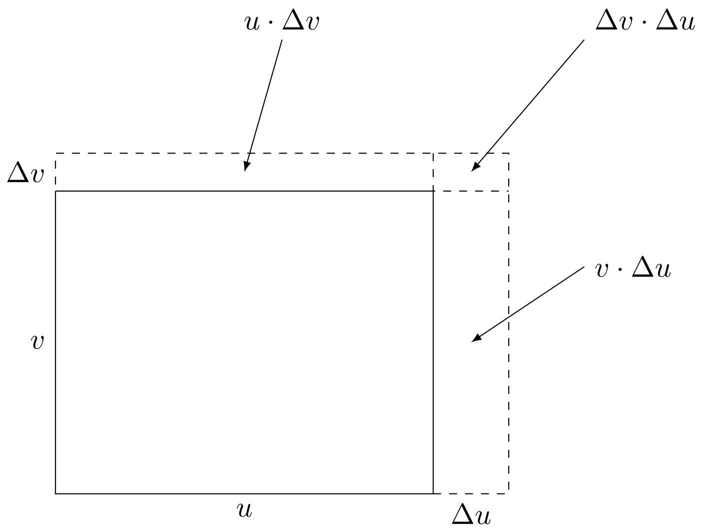
Property 13 Let $f(x)$ and $g(x)$ be two differentiable functions, then the product is differentiable and

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Proof

Let's call $f(x) = u$ and $g(x) = v$. Now, consider the product $u \cdot v$ and its variation: the product $u \cdot v$ can be interpreted as the area of a rectangle with sides u and v.

and when x varies to $x + dx$, u varies to $u + \Delta u$ and v varies to $v + \Delta v$.



Then $u \cdot v$ varies to $v \cdot u + v \cdot \Delta u + \Delta v \cdot u + \Delta v \cdot \Delta u$ hence

$$\Delta(u \cdot v) = v \cdot \Delta u + \Delta v \cdot u + \Delta v \cdot \Delta u$$

Divide the expression above by dx

$$\frac{\Delta(u \cdot v)}{dx} = v \cdot \underbrace{\frac{\Delta u}{dx}}_{\simeq u'} + \underbrace{\frac{\Delta v}{dx}}_{\simeq v'} \cdot u + \underbrace{\frac{\Delta v}{dx}}_{\simeq v'} \cdot \Delta u$$

The last term is ultraclose to zero: recall that $\Delta u \simeq 0$ because u is continuous, so $v'\Delta u \simeq 0$ by rule 2, page 20 (1). $\square$

Exercise 35

Using the derivatives of $f : x \mapsto x^2$ and $g : x \mapsto x^3$, calculate the derivative of $h : x \mapsto x^5$ $(= x^2 \cdot x^3)$.

Exercise 36

Let c be a constant, considered as a constant function. What is Δc?

Complete the proof of the following:

Property 14 Let c be a constant. Then

$$c' = 0$$

Property 15 For $n \in \mathbb{N}$ If $f : x \mapsto x^n$ then $f'(x) = nx^{n-1}$

Proof
This property is for all values of n,
It is of course impossible to prove all cases. We prove by induction.

> If
>
> (1) The property holds for $n = 0$,
>
> (2) Assuming the property holds for n greater than 0, we can prove that it also holds for $n+1$,
>
> then the property holds for all $n \in \mathbb{N}$.

A proof that this method of proof is valid can be given by contradiction.

Assume (1) and (2) have been checked for a given property P, but that there is a value m such that the property does not hold. Then $m \geq 1$ since that the property has been proven for $n = 0$. Since we are in the natural numbers, every set has a smallest number, so the set of values for which the property does not hold has a smallest value, call it p.

Then the property holds for $p - 1 \geq 0$. But since (2) has been proven, we can show that the property holds for $p - 1 + 1$ so the property holds for p: a contradiction. Therefore there can be no number for which the property does not hold.

Back to the proof of property 15:

We have already noticed that the property holds for $n = 1$, $n = 2$ and $n = 3$, i.e. $(x^2)' = 2x$, $(x^3)' = 3x^2$ and $x' = 1$.

Assume it holds for n, then

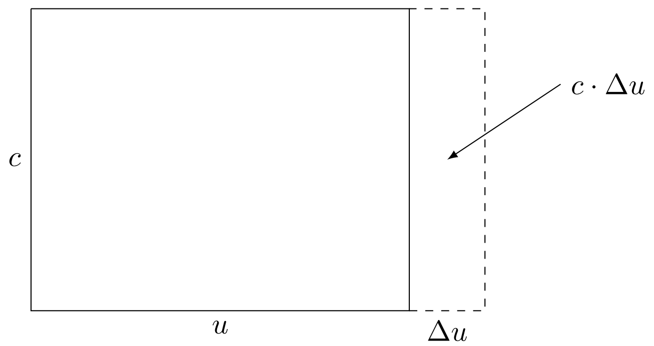
$$(x^{n+1})' = (x^n \cdot x)' = (x^n)' \cdot x + x^n \cdot x' = n \cdot x^{n-1}x + x^n = (n+1)x^n$$

so the property holds for $(n+1)$ therefore it holds for all $n \in \mathbb{N}$. $\square$

> **Property 16** Let c be a constant and $f(x)$ a differentiable function. Then $c \cdot f(x)$ is differentiable and
> $$(c \cdot f(x))' = c \cdot f'(x)$$

Proof

Let's call $f(x) = u$ and consider the product $c \cdot u$ for constant c and differentiable function u. When x varies to $x + dx$, u varies to $u + \Delta u$ and c remains constant.



The product $c \cdot u$ varies from $c \cdot u$ to $c \cdot u + c \cdot \Delta u$, hence

$$\Delta(c \cdot u) = c \cdot \Delta u$$

Divide the expression above by dx

$$\frac{c \cdot \Delta u}{dx} = c \cdot \frac{\Delta u}{dx} \simeq c \cdot u'$$

$\square$

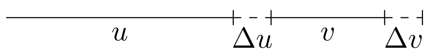
Sum and difference

Property 17 Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$(f(x) + g(x))' = f'(x) + g'(x)$$

Proof

Again, let's call $f(x) = u$, $g(x) = v$ and consider the sum $u + v$. When x varies to $x + dx$, u varies to $u + \Delta u$ and v varies to $v + \Delta v$



Then

$$\Delta(u + v) = \Delta u + \Delta v$$

Divide the expression

$$\frac{\Delta u + \Delta v}{dx} \simeq u' + v'$$

$\square$

Exercise 37

Find the derivatives of $h : x \mapsto x^3 + x^2$ and $k : x \mapsto 5x^3 - 7x^2$

Composition

Property 18 (Chain Rule)
Let f and g be real functions such that g is differentiable at x and f is differentiable at $g(x)$. The the function $f \circ g$ is differentiable at x and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof

Let's call $f(y) = u$, $g(x) = v$. If u' exists, we have (as usual)

$$u' \simeq \frac{\Delta u}{dx}$$

where u depends on v

If $\Delta v \neq 0$, then

$$u' \simeq \frac{\Delta u}{dx} = \frac{\Delta u}{\Delta v} \cdot \frac{\Delta v}{dx} \simeq u' \cdot v'$$

and the property holds.

Otherwise, if $\Delta v = 0$, then $v' = 0$. Furthermore from x to $x + dx$, v does not change its value since $\Delta v = 0$ hence u does not change its value so $\Delta u = 0$. Hence $(u \circ v)' = 0 = u' \cdot \underbrace{v'}_{=0}$ □

Using the concept of differential (page 47), the chain rule can be shown (for y as function of x and z as function of y):

$$dz = z' \cdot dy = z' \cdot y' \cdot dx$$

hence

$$\frac{dz}{dx} = z' \cdot y'$$

Note that this does not require to separate the case when $dy = 0$.

Exercise 38

Give the derivatives of the following functions and find the zeroes of these derivatives:

(1) $f : x \mapsto (x^3 + 2x)^4$

(2) $g : x \mapsto (5x^3 + 3x^2)^{13}$

Exercise 39

Use $(\sqrt{x})^2 = x$ and property 18, page 56 to find the derivative of $y = \sqrt{x}$ (for $x > 0$) – assuming it exists.

Exercise 40

Give the derivatives of the following functions:

(1) $f : x \mapsto (\sqrt{x} + 1)^4$

(2) $g : x \mapsto \sqrt{5x^3 + 3x^2}$

(3) $h : x \mapsto \sqrt{x^2}$

(4) $i(x) = \sqrt{3x^3 + 2x + 1}$

(5) $j(x) = (x^2 + 3)^5$

(6) $k(x) = (ax + b)^n$

(7) $l(x) = \sqrt{x^3 + 1}$

Exercise 41

Use the definition of the derivative to find $f'(x)$ for $f : x \mapsto \frac{1}{x}$

Exercise 42

Use the previous exercise and the chain rule to find the derivative of $\frac{1}{f(x)}$, assuming that $f(x) \neq 0$ and that $f'(x)$ exists.

Quotient

Exercise 43

Write $\frac{u}{v} = u \cdot \frac{1}{v}$ and use exercises 42 and the chain rule to prove the following:

> **Property 19** Let f and g be two real functions differentiable at a and $g(a) \neq 0$. Then the function $\dfrac{f}{g}$ is differentiable at a and
>
> $$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g^2(a)}$$

Exercise 44

Show that for $m \in \mathbb{Z}$

$$(x^m)' = m \cdot x^{m-1}$$

Exercise 45

Find the slope of $x \mapsto \dfrac{x^2 - 2x}{x^3 + x^2}$ at $x = 1$.

Exercise 46

Find the derivative of

$$f : x \mapsto \frac{x}{x^2 + 1}$$

Practice exercise PE16 Answer page 125

Differentiate the following for general x:

(1) $f : x \mapsto 5x^4 + x^3 - 2x^2 + 25$

(2) $g : x \mapsto 5\sqrt{3}\, x^2 - 100$

(3) $h : x \mapsto \dfrac{x^2 + 2x - 1}{x^3 - 5}$

(4) $j : x \mapsto 5x^4 + \dfrac{1}{3x^2 - 2x + \pi}$

(5) $k : x \mapsto (5x + 2) \cdot \dfrac{1}{5x + 2}$

(6) $l : x \mapsto \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4}$

(7) $m : x \mapsto \dfrac{1 + x}{1 + \frac{1+x}{x^2}}$

Practice exercise PE17 Answer page 125

Sketch the curve of $y = -(x-3)(x+1)(x-1)$

Practice exercise PE18 Answer page 126

Let $y = \dfrac{10x}{x^2+1}$. Sketch the curve and give the equation of the line tangent to the curve at $x = 3$.

Practice exercise PE19 Answer page 126

Consider each of the following as a function f, find the corresponding derivative function f'.

(1) $x^3 + x^2 + 2x - 4$

(2) $-x^3 + 2x^2 - 2x + 1$

(3) $\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$

(4) $\frac{1}{3}(x-2)^3$

(5) $\dfrac{x^2}{x+2}$

(6) $x - 1 + \dfrac{9}{x+1}$

(7) $\dfrac{4x^2+4x+5}{4x+2}$

(8) $\dfrac{-x^2-2x-1}{x+3}$

(9) $|x-2|$

(10) $\dfrac{x^2}{|x|+2}$

(11) $x + 2 - \dfrac{1}{x+1}$

(12) $|x^3 - 6x^2 + 11x - 6|$

Exercise 47

Find the derivative of the following functions. Since they are piecewise defined, the answer will be in 3 parts – one special point is the meeting point for both rules.

(1)

$$f : x \mapsto \begin{cases} x^2 & \text{if } x \geq 1 \\ 2x - 1 & \text{if } x < 1 \end{cases}$$

(2)

$$g : x \mapsto \begin{cases} x^2 & \text{if } x > 2 \\ x + 2 & \text{if } x \leq 2 \end{cases}$$

(3)

$$h : x \mapsto \begin{cases} x^2 & \text{if } x \geq 3 \\ 2x & \text{if } x < 3 \end{cases}$$

Exercise 48

Using that $(x^{\frac{1}{n}})^n = x$, find the derivative of $y = x^{\frac{1}{n}}$

This shows that the rule in property 15 holds also for rational n[1].

Exercise 49

Use $\mid x \mid = \sqrt{x^2}$ to find an expression for the derivative of $\mid x \mid$

Practice exercise PE20 Answer page 127

Let

$$f : x \mapsto \frac{1}{3}x^3 + \frac{7}{2}x^2 + 12x$$

Calculate its derivative, find where the derivative is positive, where it is negative and where it is equal to zero.

Calculate the roots and intercept (if any) of f and sketch the graph of f.

Practice exercise PE21 Answer page 127

(1) $f : x \mapsto 2x^2 - 4x + 5$

(2) $g : x \mapsto \dfrac{x^3 + 2x}{7}$

For f, give the equation the line tangent to the curve at $x = -2$

For g, give the equation the line tangent to the curve at $x = 1$

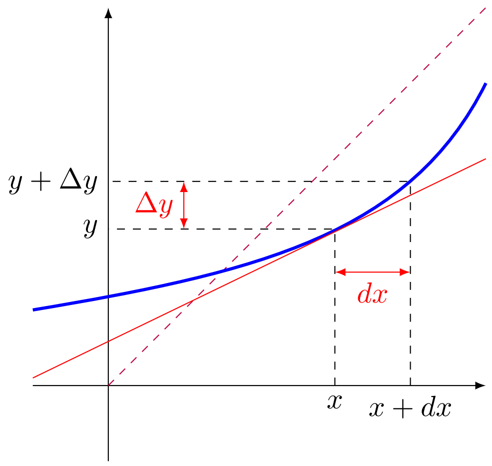
Derivative of the inverse function

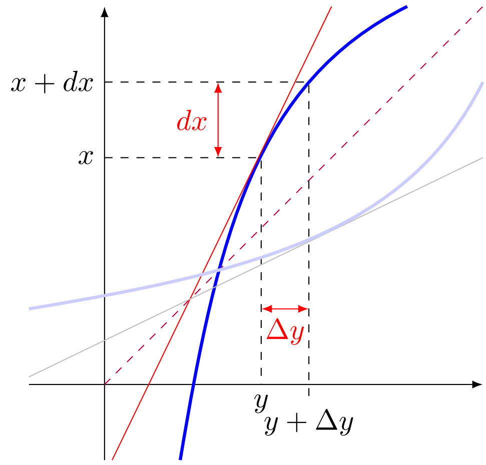
Let f be a function. Recall that the inverse function of f, if it exists, is written f^{-1} and is such that $f^{-1}(f(x)) = x$.

⚠ $f^{-1}(x)$ is <u>not</u> $\dfrac{1}{f(x)}$

A function has an inverse if the image of its curve by a symmetry through the $y = x$ axis is the curve of a function.

[1] For now, we assume without proof, that the nth root is differentiable.





A continuous function has an inverse if and only if it is everywhere strictly increasing or strictly decreasing. If it had a maximum, there would be two points a and b with $f(a) = f(b) = y$ and the inverse would have $f^{-1}(y) = a$ and b, which contradicts the definition of a function. A horizontal segment would have a vertical segment as symmetry: also not the graph of a function.

Because if $f(x) = y$ is differentiable then it is continuous i.e., $\Delta y \simeq 0$. Looking at the symmetric graph one can see that if y is the independent variable with an increment $\Delta y \simeq 0$, the corresponding variation of x (as dependent variable!) must be ultrasmall (because the function is strictly increasing/decreasing.

The slope of the tangent of the inverse is the reciprocal of the slope of the original tangent:

$$\frac{dx}{\Delta y} = \frac{1}{\frac{\Delta y}{dx}} \simeq \frac{1}{y'}$$

Note that writing dy instead of Δy leads to the same result but is not absolutely correct: the variation of the functions should be used rather than the variation along their tangent lines. The derivative is the observable neighbour of the ratio $\frac{\text{vertical variation}}{\text{horizontal variation}} = \frac{\Delta y}{dx} \simeq \frac{dy}{dx}$.

Switching variables leads to $\frac{dx}{\Delta y} = \frac{dx}{y'dx + \varepsilon dx} = \frac{1}{y' + \varepsilon} \simeq \frac{1}{y'}$.

Property 20 (Derivative of the Inverse)
Let f be a function defined on an interval I with image J. If f is differentiable on I, has an inverse f^{-1}, and $f'(a) \neq 0$ for $a \in I$, then this inverse is differentiable at $b = f(a) \in J$ and

$$\frac{\Delta f^{-1}(b)}{\Delta y} = \frac{1}{f'(a)}$$

In general form:

$$\frac{\Delta f^{-1}(y)}{\Delta y} = \frac{1}{f'(x)}$$

Summary of this chapter

- $(x^m)' = mx^{m-1}$, for $m \in \mathbb{Q}$.
- $c' = 0$
- $(c \cdot f)' = c \cdot f'$
- $(f + g)' = f' + g'$
- $(f \cdot g)' = f' \cdot g + f \cdot g'$
- $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$
- $(f \circ g)' = f' \cdot g'$
- $(f^{-1})' = \frac{1}{f'}$

Chapter 7

Curve sketching – basic functions

To sketch the curve of a function on a Cartesian plane involves problem-solving abilities. The task can be carried out by taking, in sequence, the following steps:

(1) Find the domain.

(2) Find the roots and the intercept (if any).

(3) Find the asymptotes (if any).

(4) Find the derivative (if any).

(5) Find the roots of the derivative (if any).

(6) Determine the maximum and minimum values.

(7) Use this information to choose a convenient scale.

(8) Sketch the function.

The roots are also called the x-intercepts. There can be none, one, or more.

The intercept ($f(0)$) is also called the y-intercept, there can be none if the function is not defined at zero, or exactly one.

Example: Let

$$f : x \mapsto \frac{x}{x^2+1}$$

(1) The denominator is never zero ($x^2 + 1 \geq 1$) so the function's domain is $\mathbb{R}$

(2) $f(0) = 0$ (x-intercept and y-intercept)

(3) The denominator if the function $x^2 + 1 \geq 1$ hence $x/(x^2 + 1) \leq x$. Let a be observable, then $f(a) \leq a$ hence is not ultralarge, The function does not have a vertical asymptote. For the horizontal asymptote, see exercise 7, page 30. Divide numerator and denominator

by x, and consider x to be ultralarge

$$\frac{x}{x^2+1} = \frac{1}{x + \underbrace{\frac{1}{x}}_{\simeq 0}} \simeq \frac{1}{x} \simeq 0$$

There is a horizontal asymptote at $y = 0$ (on the right and on the left).

(4)

$$f'(x) = \frac{(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

(5) $f'(x) = 0 \Rightarrow x \in \{-1, +1\}$

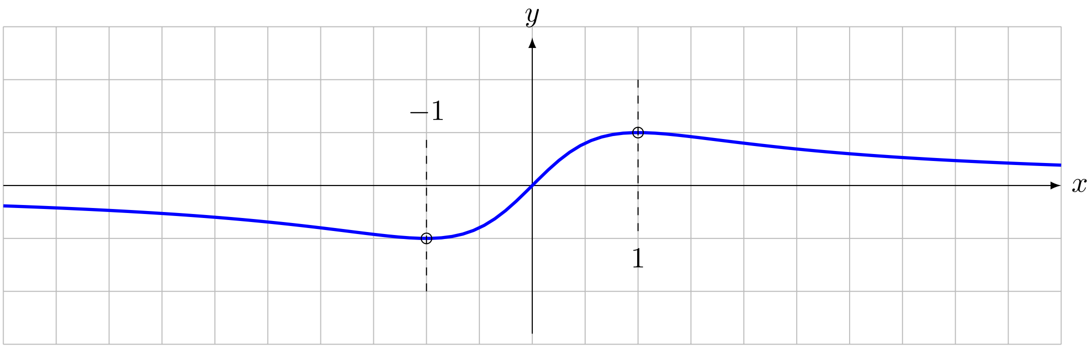
(6)

	$-\infty$		-1		0		1		∞
f	$\simeq 0$	$-$	$-$	$-$	0	$+$	$+$	$+$	$\simeq 0$
f'		$-$	0	$+$	$+$	$+$	0	$-$	
f		$\searrow$	*min*	$\nearrow$	$\nearrow$	$\nearrow$	*max*	$\searrow$	

There is a minimum at $x = -1$ and a maximum at $x = 1$.

(7) A convenient scale for the drawing can be chosen by calculating:
$f(-1) = \frac{-1}{(-1)^2+1} = -\frac{1}{2}$
$f(1) = \frac{1}{2}$

(8)



Practice exercise PE22 Answer page 128
Sketch the curves of the following functions:

(1) $f_1 : x \mapsto \dfrac{x^2}{x+2}$

(2) $f_2 : x \mapsto x - 1 + \dfrac{9}{x+1}$

(3) $f_3 : x \mapsto \dfrac{-x^2-2x-1}{x+3}$

(4) $f_4 : x \mapsto x + 3 + \dfrac{1}{2x+1}$

(5) $f_5 : x \mapsto \dfrac{x^2-4x+6}{(x-2)^2}$

(6) $f_6 : x \mapsto \dfrac{x^3-1}{x^2}$

Chapter 8

Optimisation and other problems

Exercise 50

(1) Find the slope of the curve given by $f : x \mapsto 5x^3 - 25x^2$ at $x = 3.5$

Equivalent notations:

- $f'(3.5)$
- $f'(x)\Big|_{x=3.5}$

(2) Find the equation of the line tangent to the curve at $x = 1$

Exercise 51

(1) For $f : x \mapsto x^2 + 5$ and the point $A\langle 0, 0\rangle$, what is the equation of the line passing through A, and tangent to f? Is it unique?

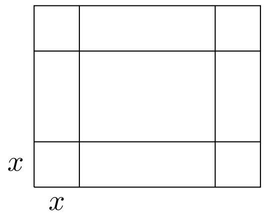
(2) If $g : x \mapsto ax^2 + b$, what values do a and b take to make $g(x)$ tangent to $t : x \mapsto 3x - 2$ at $x = 5$? What are the coordinates of the contact point?

Optimal solutions

The optimal solution for a given problem is often equivalent to finding a maximum or minimum of a function which expresses the quantity to be optimised. Bearing in mind the critical point theorem 4, page 48, the optimum could be at a non differentiable value, or looking for a maximum, one could have found a minimum, so checking is required.

Exercise 52

A factory wants to make cardboard boxes (with no top) out of sheets of 30cm $\times$ 16cm



The volume will be a function of x. The dimensions of the base are $30 - 2x$ and $16 - 2x$ (in centimetres). The height is x. What value(s) of x give(s) the maximum volume for the box?

Exercise 53

A 1l (1dm^3) milk pack is made of cardboard. Its sides can only be rectangles. The height is twice one of the other two dimensions. The area of the outside of the pack must be minimal.

What are the dimensions of the pack?

Exercise 54

Imagine you want to protect a part of a rectangular garden against a wall. You have 100m of fence. (No fence is needed against the wall.)

What is the biggest area that you can protect?

Exercise 55

A cylindrical jar has a volume defined by its radius and its height. If it contains 1l, what are the dimensions that will make it have the least outside area?

Exercise 56

Find the length and width of the rectangle inscribed within the ellipse given by the formula $4x^2 + y^2 = 16$ (sides parallel to the coordinate axes) such that its area is maximal.

Exercise 57

Let $\mathcal{P}$ be the parabola given by $x \mapsto x^2$ and A be the point $\langle 0, 5\rangle$

Find the point(s) on the parabola $\mathcal{P}$ such that its (their) distance to A is minimal.

Price setting

The goal of the price-setting process shown here is to set profit-maximizing prices.

It usually involves assuming an initial price (and/or profit) and setting an expected sales volume. Then the marketing strategist predicts how a change in price will affect sales volume. Price setting is then the process of finding the best price.

Exercise 58

A hardware store has 300 lawnmowers. The manager assumes that at 200 € each, they will all be sold.

The manager also assumes that for each increase by 10 €, there will be 2 less sold.

Determine the retail price that will ensure the maximal income. (Which requires to write the equation for the profit depending on the number of times 10 € have been added to the price.)

Exercise 59

Now the hardware store must pay the lawnmowers and has administrative expenses which decrease with the number of lawnmowers.

The price of a lawnmower for the hardware store is 120 € and the administrative expenses are assumed to be $(0.5 + \frac{1}{n})$ €, where n is the number of lawnmowers that are sold.

Determine the price which will maximise the profit of the hardware store with these new conditions.

Chapter 9

Areas

Exercise 60

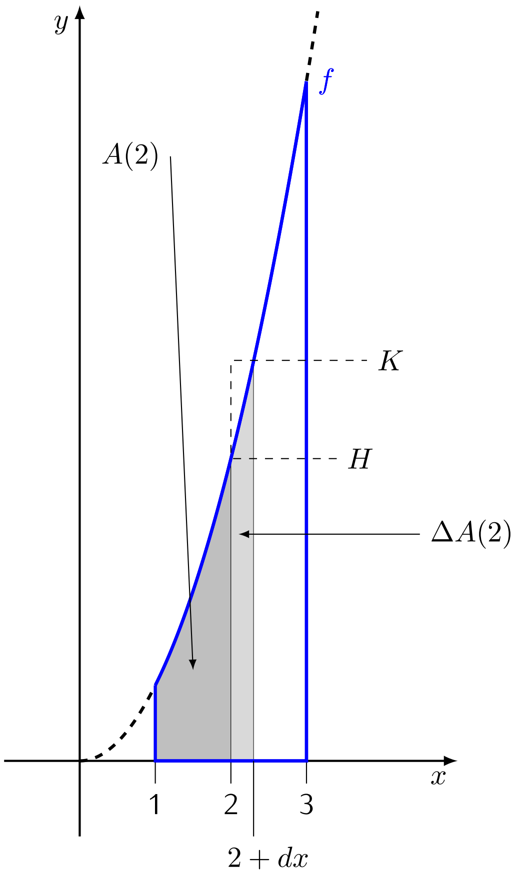
Find the derivative of $f : x \mapsto \frac{x^3}{3}$.

Same question for $g : x \mapsto \frac{x^3}{3} + 5$

Consider

$$f : x \mapsto x^2$$

We would like to calculate the area under the curve, above the x-axis, between $x = 1$ and $x = 3$



(of course dx is drawn much too large so as to understand where it is.)

There is a difficulty: we know formulae for calculating areas of rectangles, triangles, circles and some other figures, but not for an arbitrary curved surface.

The great idea of analysis is to discover functions by observing their variation.

For this we consider first the area up to 2 and its variation to $2+dx$. The area of the curved slice $\Delta A(2)$ is between the area of the rectangle whose base is dx and whose height is H and the rectangle whose base is dx and whose height is K.

If $dx > 0$ we have $f(x+dx) > f(x)$ and

$$H \cdot dx = f(2) \cdot dx < \Delta A(2) < f(2+dx) \cdot dx = K \cdot dx$$

and dividing all terms by dx

$$f(2) < \frac{\Delta A(2)}{dx} < f(2+dx)$$

and we conclude that

$$f(2) \simeq \frac{\Delta A(2)}{dx}$$

The conclusion is the same for a variation on the left:

$$f(2-dx) \cdot dx < \Delta A(2) < f(2) \cdot dx$$

which also leads to $f(2) \simeq \frac{\Delta A(2)}{dx}$

Therefore $A'(2) = f(2)$ and in general we will have $A'(x) = f(x)$. Using results of exercise 60, it is possible to check that $A(x) = \dfrac{x^3}{3}$ but $\dfrac{x^3}{3} + k$ also satisfies the requirement. For both cases, in fact, $A'(x) = x^2 = f(x)$.

Our problem seems to be underdetermined: we have the unknown parameter, k. The question now is: what value does k have to take?

Let's consider $A(1)$, the area under f from 1 to 1. Surely, $A(1) = 0$, but we also know that $A(1) = \dfrac{1^3}{3} + k$, hence $0 = \dfrac{1^3}{3} + k \Rightarrow k = -\frac{1}{3}$. Now that we have determined the value of the parameter k, let's get back to our initial task and calculate the area under the curve $f(x)$ between $x = 1$ and $x = 3$; this is:

$$A(3) = \frac{3^3}{3} + k = \frac{3^3}{3} - \frac{1}{3} = \frac{8}{3}.$$

This is a first step in showing what was hinted in the introduction page 2, that the area and the slope are inverse properties.

Exercise 61

Using the same method as above, calculate the area under $f : x \mapsto x^2$ and above the x-axis, between $x = 2$ and $x = 6$. Use that $A(2) = 0$.

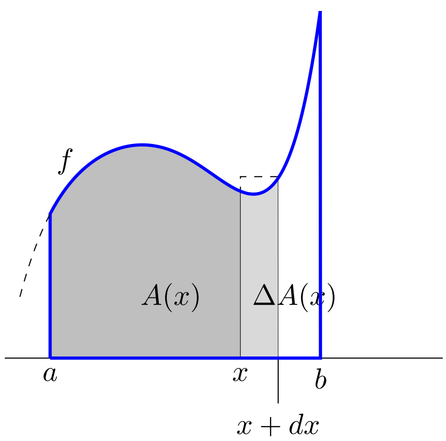
Property 21 Let f be a non-negative function continuous on $[a, b]$. Then the function

$$A : x \mapsto A(x)$$

where $A(x)$ is the area under the curve of f between a and x has the following properties

(1) $A'(x) = f(x)$, whenever $x \in [a, b]$

(2) $A(a) = 0$



Proof

f is assumed to be continuous so on any closed interval it has a maximum and a minimum (see theorem 3, page 38).

For $dx > 0$. On $[x, x + dx]$ the function reaches a maximum, $f(x_M)$, and a minimum, $f(x_m)$. Hence the slice $\Delta A(x)$ is bounded below by the rectangle $f(x_m) \cdot dx$ and above by the rectangle $f(x_M) \cdot dx$.

While it was clear for the example of the beginning of this chapter where the maximum and minimum were (because we knew the function's rule), what is done here is to show that we can find the same relation for any continuous function.

Hence

$$f(x_m) \cdot dx \leq \Delta A(x) \leq f(x_M) \cdot dx$$

then, dividing by dx we get:

$$f(x_m) \leq \frac{\Delta A(x)}{dx} \leq f(x_M)$$

And since x_m and x_M are in $[x, x + dx]$ we have $x_m \simeq x \simeq x_M$
The statement is about f, a and x.
$x_m \simeq x \Rightarrow f(x_m) \simeq f(x)$
$x_M \simeq x \Rightarrow f(x_M) \simeq f(x)$
We conclude that

$$\frac{\Delta A(x)}{dx} \simeq f(x)$$

By taking $dx < 0$ we notice that the area decreases and the the inequalities are reversed, hence, not depending on the choice of dx (provided it is ultrasmall) we have

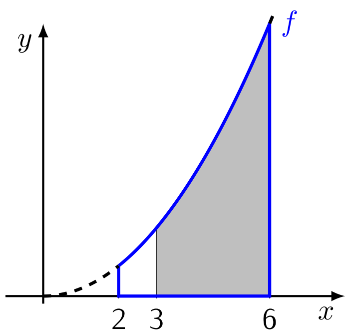
$$\frac{\Delta A(x)}{dx} \simeq f(x) \Rightarrow A'(x) = f(x)$$

Note that if f is a negative function, then $\Delta A(x) \simeq f(x) \cdot dx < 0$: the method described above for the area between the function and the x-axis will produce a negative number.

$A(a) = 0$ by the definition that it is the area between a and a □

In exercise 61, page 71, an area is found using a function A with a constant given by the condition $A(2) = 0$. Now we would like to have the area under $f : x \mapsto x^2$ between $x = 3$ and $x = 6$:

The figure will look like the following (vertical scale and horizontal scale not the same)



We notice that the area in gray can be computed from what we already know by exercise 61 and the corresponding area function $A : x \mapsto \frac{x^3}{3} - 4$ by considering it is the area between 6 and 2 minus the area between 3 and 2, all of which can be computed using A as before, since the condition $A(2) = 0$ is the same for both areas.

$$\text{Area from 3 to 6} = A(6) - A(2) - [A(3) - A(2)] = A(6) - A(3) \qquad (*)$$

$A(2)$ which gave the condition $k = -4$ disappears.
We compute the values : $A(6) = \frac{6^3}{3} - 4$ and $A(3) = \frac{3^3}{3} - 4$ so

$$A(6) - A(3) = \frac{6^3}{3} - \frac{3^3}{3}$$

the constant $k = -4$ cancels.

One can notice that if $B(x) = A(x) + k$ for any constant k, the result will be the same: the constant cancels in any computation of the form $B(b) - B(a)$.

Notation

$$f(b) - f(a) \text{ is written } f(x)\Big|_a^b$$

so $A(b) - A(a) = A(x)\Big|_a^b$

> **Definition 18 (Integral)** The generalisation of the concept of area — such that when the function is positive on $[a, b]$, this produces a positive value, and when the function is negative, this produces a negative value — is the **integral** of f between a and b, denoted by
> $$\int_a^b f(x) \cdot dx$$

The reason why the integral notation uses a product $f(x) \cdot dx$ will be justified page 75.

Using the notation introduced above and recalling the method used to calculate the areas under the function curves, we can write:

$$\int_a^b f(x) \cdot dx = A(b) - A(a)$$

or, equivalently:

$$A(x)\Big|_a^b = \int_a^b f(x) \cdot dx \qquad \text{where } A'(x) = f(x).$$

This is the a number which represents the generalised area, the integral, of f between a and b.

So the area denoted by (*) can be written

$$\int_3^6 x^2 \cdot dx = \frac{x^3}{3}\Big|_3^6$$

Antiderivative

> **Definition 19 (Antiderivative)** An antiderivative of a function f is a function A such that $A'(x) = f(x)$, and it is denoted by: $A = \int f(x) \cdot dx$

By property 11, page 50, the antiderivative is unique up to an additive constant.

Exercise 62

Find antiderivatives for the following:

(1) $x \mapsto 3x$

(2) $x \mapsto x^2$

(3) $x \mapsto 5$

(4) $t \mapsto 3t + 5$

(5) $u \mapsto u^2 + 3u + 5$

(6) $v \mapsto v^3$

(7) $x \mapsto \frac{1}{\sqrt{x}}$

Check your results by differentiating them.

Exercise 63

Newton assumed that objects fall to the ground with a constant acceleration, denoted by g. Given such an acceleration, how can one find the equation of the distance travelled by a falling object with respect to time?

Exercise 64

Using $A' = f$ and $A(a) = 0$:

(1) Calculate the area between the curve and the x-axis for $y = x^2$ from $x = -5$ to $x = 5$

(2) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = 0$ to $x = 3$

(3) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -2$ to $x = 0$

(4) Calculate the area between the curve and the x-axis for $y = x^3$ from $x = -10$ to $x = 10$

Exercise 65

Calculate the area between $y = 5x^4 - 3x^3 + 2x^2 - 10$ and the x-axis from $x = -1$ to $x = 1$

Fundamental theorem of calculus

All these results put together yield:

Theorem 7 (Fundamental theorem of calculus)
Let f be a function continuous on $[a, b]$

(1) Then

$$F(x) = \int_a^x f(t) \cdot dt$$

is an antiderivative of f on $]a, b[$ and is the only one satisfying $F(a) = 0$

(2) Let F be an antiderivative of f on $]a, b[$. Then

$$\int_a^b f(x) \cdot dx = F(b) - F(a)$$

This is essentially re rephrasing of property 21 using definitions 18 and 19.

The integral as ultralarge sum

The $\int$ symbol is an elongated S and stands for the Latin word "summa": a sum, since it can also be shown that instead of finding the area as a variation, it is a $\int$um of $\int$lices.

Exercise 66

Consider the variation of F between a and b.
Let $N \in \mathbb{N}$ such that $1/N \simeq 0$ and $dx = \frac{b-a}{N}$ and $x_k = a + k \cdot dx$, for k between 0 and N.
Then clearly, we have

$$F(b) - F(a) = \sum_{k=0}^{N-1} \Delta F(x_k)$$

Here the observability is determined by f, a, b – not necessarily any given x_j!

(1) On each interval $[x_k, x_{k+1}]$ (which is also in the form $[x_k, x_k + dx]$) there is a c such that

$$F(x_k + dx) - F(x_k) = f(c) \cdot dx$$

This is due to the mean value theorem, (theorem 6, page 49).

(2) Explain why we have $f(c) \simeq f(x_k)$

(3) Conclude by explaining why:

$$\sum_{k=0}^{N-1} F(x_k + dx) - F(x_k) = \sum_{k=0}^{N-1} f(x_k) \cdot dx + \sum_{k=0}^{N-1} \varepsilon_k \cdot dx$$

(4) The part

$$\sum_{k=0}^{N-1} \varepsilon_k \cdot dx$$

is ultrasmall. To prove this, let $\varepsilon = \max\{|\varepsilon_k| \mid 0 \leq k \leq N\}$

Complete the proof of this claim.

so

$$\simeq \sum_{k=0}^{N-1} f(x_k) \cdot dx$$

Hence, the global variation of F between a and b is, up to an ultrasmall value, the sum of $F'(x_i) \cdot dx$ provided F' is continuous on $[a, b]$

But then since $F(b) - F(a)$ is observable (by closure), the global variation can be defined as the observable neighbour of $\sum_{k=0}^{N-1} f(x_k) \cdot dx$

If bounds are given, the integral represents a value: it is a **definite integral**. If no bounds are given, it represents an antiderivative: it is an **indefinite integral**.

Exercise 67

Show that for a definite integral, it does not matter which antiderivative is chosen.

Since the antiderivative is unique up to an additive constant (property 11, page 50), this means that it does not depend on the value of the constant.

The results obtained in this section can be formalized in an alternative definition of integral.

Definition 20 (Integral as an ultralarge sum) Let f be a function (continuous or piecewise continuous) defined on an interval $[a, b]$, the **integral** $\int_a^b f(x)dx$ is the observable neighbour (if it exists) of

$$\sum_{k=0}^{N-1} f(x_k) \cdot dx$$

(with $dx = \frac{b-a}{N}$), provided also that this value does not depend on the ultralarge number N.

Definitions 18, page 73 and 20 are in fact equivalent for continuous functions; definition 18 is simpler and will be usually used in the following.

Exercise 68

A constant function $f : x \mapsto C$ from a to b defines a rectangle. Check that the area under f is the "usual" formula: $(b - a) \cdot C$

Exercise 69

The function $y = x$ defines a triangle for x between 0 and 4. Show that the area of the triangle from 0 to a yields the "usual" result for the area of a triangle.

Exercise 70

Sketch the curve of $f : x \mapsto x^2$ and $g : x \mapsto x^3$. Calculate the points where $f(x) = g(x)$

Calculate the geometric area of the closed surface between the two curves.

Integration rules

Property 22 (Linearity of the integral) Let f and g be real functions continuous on $[a, b]$. Let λ be a real number. Then

(1)
$$\int_a^b (\lambda \cdot f(x)) \cdot dx = \lambda \cdot \int_a^b f(x) \cdot dx$$

(2)
$$\int_a^b (f(x) + g(x)) \cdot dx = \int_a^b f(x) \cdot dx + \int_a^b g(x) \cdot dx$$

These are simply the converse rules of linearity of the derivative.

Property 23 (Additivity of the integral) Let f be a real function continuous on $[a, c]$ and $b \in [a, c]$. Then

$$\int_a^b f(x) \cdot dx + \int_b^c f(x) \cdot dx = \int_a^c f(x) \cdot dx$$

Exercise 71

For each of the following functions, find the general form of the antiderivative:

(1) $f : x \mapsto 8\sqrt{x}$

(2) $f : t \mapsto 3t^2 + 1$

(3) $f : t \mapsto 4 - 3t^3$

(4) $f : s \mapsto 7s^{-3}$

(5) $f : x \mapsto (x - 6)^2$

(6) $f : y \mapsto y^{\frac{3}{2}}$

(7) $f : x \mapsto |x|$

(8) $f : u \mapsto u^2 + u^{-2}$

(9) $f : x \mapsto 4$

(10) $f : t \mapsto t$

(11) $f : z \mapsto \dfrac{2}{z^2}$

Check your results by differentiating them.

Exercise 72

(1) If $F'(x) = x + x^2$ for all x, find $F(1) - F(-1)$

(2) If $F'(x) = x^4$ for all x, find $F(2) - F(1)$

(3) If $F'(t) = t^{\frac{1}{3}}$ for all t, find $F(8) - F(10)$

Property 24 (Integration with inside derivative) Let f and g be real functions differentiable on $[a, b]$ such that f' and g' are continuous on $[a, b]$. Then

$$\int_a^b f'(g(x)) \cdot g'(x) \cdot dx = f(g(x))\Big|_a^b$$

Exercise 73
Prove property 24.

Property 25 (Integration by parts) Let f and g be real functions continuous on $[a,b]$ such that f' and g' are continuous on $[a,b]$. Then

$$\int_a^b f'(x)\cdot g(x)\cdot dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f(x)\cdot g'(x)\cdot dx$$

Proof
This is the converse of the product rule $(u\cdot v)' = u'v + uv'$

$$(u\cdot v)' = u'v - uv'$$

hence

$$\int_a^b (u\cdot v)'\cdot dx = \int_a^b u'v\cdot dx - \int_a^b uv'\cdot dx$$

$$u\cdot v\Big|_a^b = \int_a^b u'v\cdot dx - \int_a^b uv'\cdot dx$$

□

Integration by variable substitution

In this section, the differential notation (see page 47) and the chain rule (see page 56) are used extensively.

Consider

$$\int_a^b f(g(x))\cdot dx$$

If we write $g(x) = u$ then $\frac{du}{dx} = u'(x)$ and $dx = \frac{du}{u'}$, $f(g(x))\cdot dx$ becomes $\frac{f(u)}{u'}\cdot du$ and the bounds must be changed to a_1 and b_1 so that $a_1 = g(a)$ and $b_1 = g(b)$.

This yields:

$$\boxed{\int_a^b f(g(x))\cdot dx = \int_{a_1}^{b_1} \frac{f(u)}{u'}\cdot du.}$$

The difficulty in using this method is usually to find which variable substitution is best.

Example: Evaluate

$$\int_1^2 2x\cdot(x^2+1)^2\cdot dx$$

Considering that $2x$ is the inside derivative, the antiderivative can be seen to be $\frac{(x^2+1)^3}{3}$, and

$$\left.\frac{(x^2+1)^3}{3}\right|_1^2 = (5^3 - 2^3)/3 = 39$$

Here we consider another approach by variable substitution.
Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$ hence $dx = \frac{du}{2x}$.
Then

$$2x \cdot (x^2+1)dx = 2x \cdot u^2 \cdot \frac{du}{2x} = u^2 \cdot du$$

As for the bounds: if $x = 1$ then $u = x^2 + 1 = 2$ and if $x = 2$ then $u = 4 + 1 = 5$, hence

$$\int_1^2 2x \cdot (x^2+1)^2 \cdot dx = \int_2^5 u^2 \cdot du = \left.\frac{u^3}{3}\right|_2^5$$

which gives $(125 - 8)/3 = 39$.

Example: Let

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx$$

Consider the variable change $u = 1 + \sqrt{x}$. Then $x = (u-1)^2 = g(u)$, the derivative of g is continuous. If $x = 0$ then $u = 1$ and if $x = 1$ then $u = 2$
$\sqrt{1+\sqrt{x}} = \sqrt{u}$ if $u = 1 + \sqrt{x}$)

$$dx = 2 \cdot (u-1) \cdot du$$

Replacing all terms we obtain

$$\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx = 2\int_1^2 \sqrt{u} \cdot (u-1) \cdot du = 2\int_1^2 \left(u^{3/2} - u^{1/2}\right) \cdot du$$

so that

$$2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Bigg|_1^2 = \frac{8+8\sqrt{2}}{15}$$

As g has an inverse which is $x \mapsto 1 + \sqrt{x}$ and is differentiable (except at $x = 0$), we can revert to the variable x and find an antiderivative:

$$\int \sqrt{1+\sqrt{x}} \cdot dx = \frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5 - \frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3 + C$$

Exercise 74

Calculate

$$\int_0^1 \sqrt{5x+2} \cdot dx$$

Use $u = 5x + 2$. Calculate du, change the bounds, calculate the integral.
Same integral. Use $v = \sqrt{5x+2}$

Exercise 75

Use variable substitution to evaluate the following:

(1) $\displaystyle\int_0^{10} \frac{1}{(2x+2)^2} \cdot dx$

(2) $\displaystyle\int (3-4z)^6 \cdot dz$

(3) $\displaystyle\int_{-1}^{1} 2t\sqrt{1-t^2} \cdot dt$

(4) $\displaystyle\int_a^b \sqrt{3y+1} \cdot dy$

(5) $\displaystyle\int \frac{4y}{(2+3y^2)^2} \cdot dy$

(6) $\displaystyle\int_{-2}^{2} x(4-5x^2)^2 \cdot dx$

(7) $\displaystyle\int (1-x)^{\frac{3}{2}} \cdot dx$

Practice exercise PE23 Answer page 128

(1) $\displaystyle\int_0^1 \frac{u}{\sqrt{1-u^2}} \cdot du$

(2) $\displaystyle\int_1^2 \frac{u}{\sqrt{1-u^2}} \cdot du$

(3) $\displaystyle\int_0^1 \sqrt{1+\sqrt{x}} \cdot dx$

(4) $\displaystyle\int_0^{10} t(t^2+3)^{-2} \cdot dt$

(5) $\displaystyle\int_{\sqrt{6}}^{5} x(x^2+2)^{\frac{1}{3}} \cdot dx$

(6) $\displaystyle\int_{-1}^{1} \frac{x^2}{(4-x^3)^2} \cdot dx$

(7) $\displaystyle\int_1^2 \frac{1}{t^2\sqrt{1+\frac{1}{t}}} \cdot dt$

Practice exercise PE24 Answer page 128

Find the antiderivatives of the following functions:

- $f_a : x \mapsto 5x^4 - 2x + 4$
- $f_b : x \mapsto x^3 - 5x^2 + 3x - 2$
- $f_c : x \mapsto 2x - 1$
- $f_d : x \mapsto \dfrac{5}{4}x^4 - \dfrac{3}{4}x^2 + \dfrac{5}{2}x + \dfrac{3}{2}$
- $f_e : x \mapsto 2x + 1 - \dfrac{1}{x^2}$
- $f_f : x \mapsto 3 + \dfrac{2}{x^2} - \dfrac{5}{x^3}$
- $f_g : x \mapsto x^3 + \dfrac{1}{x^2}$
- $f_h : x \mapsto \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$
- $f_i : x \mapsto \dfrac{1}{\sqrt{x}} + \sqrt{x}$
- $f_j : x \mapsto (x+1)^2$
- $f_k : x \mapsto 15(3x-2)^4$
- $f_l : x \mapsto (2x+1)^3$
- $f_m : x \mapsto (3-x)^{11}$
- $f_n : x \mapsto (3-4x)^4$
- $f_o : x \mapsto \sqrt{3x-2}$
- $f_p : x \mapsto \dfrac{1}{\sqrt{x-1}}$
- $f_q : x \mapsto 4x(3-x^2)^5$
- $f_r : x \mapsto (2x-3)(x^2-3x+1)^4$

- $f_s : x \mapsto (3x^2-4x+1)(x^3-2x^2+x+3)^2$
- $f_t : x \mapsto (4x^2-5x)^2(16x-10)$
- $f_u : x \mapsto (3x-1)(3x^2-2x+5)^3$
- $f_v : x \mapsto \dfrac{2x}{(x^2+1)^2}$
- $f_w : x \mapsto \dfrac{2x+1}{(x^2+x+3)^2}$
- $f_x : x \mapsto x\sqrt{x^2+1}$
- $f_y : x \mapsto \dfrac{3x^2}{\sqrt{9+x^3}}$
- $f_z : x \mapsto (3x^2+1)\sqrt{x^3+x+2}$

Applications of the integral

Exercise 76

In the following problems an object moves along the y axis. Its velocity varies with respect to the time. Find how far the object moves between the given times t_0 and t_1

(1) $v = 2t+5$ $\quad t_0 = 0 \quad t_1 = 2$

(2) $v = 4-t$ $\quad t_0 = 1 \quad t_1 = 4$

(3) $v = 3$ $\quad t_0 = 2 \quad t_1 = 6$

(4) $v = 3t^2$ $\quad t_0 = 1 \quad t_1 = 3$

(5) $v = 10t^{-2}$ $\quad t_0 = 1 \quad t_1 = 100$

Mean value of a function

The mean value is unambiguous when we consider n points, where n is a positive integer. We now show that defining the mean value of a continuous function on $[a, b]$ as

$$\frac{1}{b-a}\int_a^b f(x)\cdot dx$$

is a natural extension of this concept.

Consider a continuous function f and the interval $[a, b]$. These determine the observability. Let N be a positive ultralarge integer. Let $dx = (b-a)/N$ and $x_i = a+i\cdot dx$, for $i = 0, \ldots, N-1$. Then the mean value of the function can be approximated by the mean value of the N points $f(x_i)$, $i = 0, \ldots, N-1$. But

$$\frac{\sum_{i=0}^{N-1} f(x_i)}{N} = \frac{dx}{b-a}\sum_{i=0}^{N-1} f(x_i) = \frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i)\cdot dx \simeq \frac{1}{b-a}\int_a^b f(x)\cdot dx$$

since f is continuous on $[a, b]$

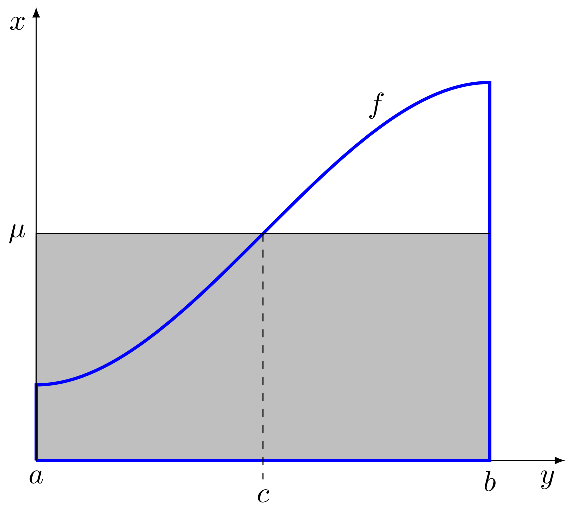
The mean value is the part of this number which is observable i.e., the integral. We therefore define:

Definition 21 The **mean value** of a function f continuous on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\cdot dx$$

The mean value is a number μ such that the area under the curve is equal to $\mu\cdot(b-a)$, i.e., the height of a rectangle of basis $(b-a)$ whose (oriented) area is equal to the integral.

In the following figure, the gray area is equal to the area bounded by the blue line.



Property 26 If f is a function continuous on $[a, b]$, then there exists a point $c \in [a, b]$ such that $f(c)$ is the mean value of the function on $[a, b]$

Note that property 26 is a restatement of the mean value theorem, for the antiderivative of f. When we claim that there is a $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\cdot dx$$

we are in fact asserting that there is a $c \in [a, b]$ such that

$$f(c)\cdot(b-a) = \int_a^b f(x)\cdot dx = F(b) - F(a)$$

and as $F'(x) = f(x)$, we conclude that there is a $c \in [a, b]$ such that $F'(c)\cdot(b-a) = F(b) - F(a)$

Exercise 77

Calculate the mean value of $x \mapsto x^2$ on $[-4, 4]$

Exercise 78

Calculate the mean value of $x \mapsto x^3$ on $[-4, 4]$.

Exercise 79

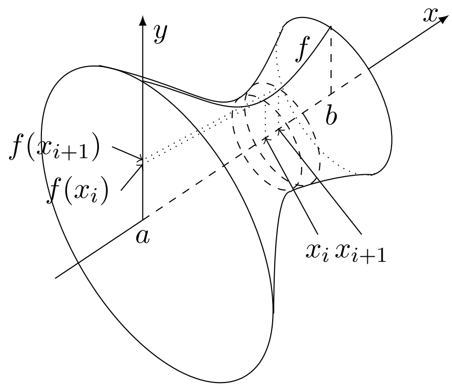
Let $f : x \mapsto x^2$ and the interval $[0, t]$. Find the value of t such that the mean value of f over the interval is equal to π.

Exercise 80

An object falling on Earth satisfies the equation $d(t) = \frac{1}{2}gt^2$ where $g \approx 9.81[\frac{\mathrm{m}}{\mathrm{s}^2}]$, t is the time in seconds and $d(t)$ is the vertical distance in meters.

If an object falls for 10s, what is its average distance from its initial point?

Solid of Revolution



Exercise 81

An area is calculated by approximating the surface by ultrasmall rectangles. To find the formula for the volume of a solid of revolution, proceed in the same manner: consider that the solid is ultraclose to an ultralarge number of ultrathin disks. Find the formula for the volume of a solid of revolution given by a function f.

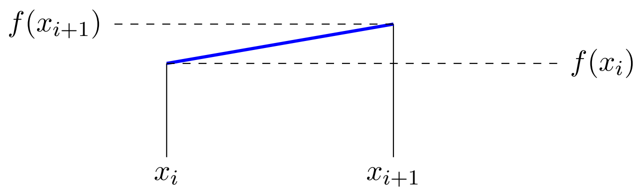
Exercise 82

Evaluate the volume of the solid of revolution of $y = \dfrac{1}{x}$ around the x-axis between $x = 1$ and $x = 10$.

Arc length

Definition 22 Let $f : [a, b] \to \mathbb{R}$ be differentiable with a continuous derivative. Then the graph of f has length

$$L = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx$$



Justification: If the function is differentiable, the function is indistinguishable from the hypotenuse of a right-angled triangle whose catheti are dx and Δf.

Assume $dx > 0$. The variation of length of this line is

$$\Delta L = \sqrt{dx^2 + (\Delta f(x))^2} = \sqrt{dx^2 \left(1 + \frac{(\Delta f(x))^2}{dx^2}\right)} = \sqrt{1 + \frac{(\Delta f(x))^2}{dx^2}} \cdot dx$$

hence

$$\frac{\Delta L(x)}{dx} = \sqrt{1 + \frac{(\Delta f(x))^2}{dx^2}} \simeq \sqrt{1 + f'(x)^2}$$

Exercise 83

Find the lengths of the following curves:

(1) $y = 2x^{3/2} \quad 0 \leq x \leq 1$

(2) $y = \dfrac{2}{3}(x+2)^{\frac{3}{2}} \quad 0 \leq x \leq 3$

Summary of this chapter

The derivative defines the slope at a given point. This is a local property. It is determined by first finding an approximation on an ultrasmall interval.

The integral defines the area under a function on an interval. This is a global property. It is determined by first dividing the interval into an ultralarge numbers of ultrasmall pieces to find an approximation.

(Similarly, the intermediate value theorem (page 37) and the extreme value theorem (page 38) are about the behaviour of a function on an interval and are proven by first dividing the interval into an ultralarge number of pieces.)

Chapter 10

Trigonometry and circular functions

Exercise 84

Given that the circumference of the Earth is (theoretically) 40 000km, what is the radius of the Earth?

Exercise 85

If a bridge were built at a constant height of 10m all around the Earth's equator – with its theoretical measure of 40 000km – how much longer than the equator would the bridge be?

Angles

An angle is defined by two lines meeting at a point called **the vertex** of the angle. The angle can be regarded as the measure of the rotation involved in moving from one line to coincide with the other line.

The rotation is measured by drawing a circle centred at the vertex and observing what amount of the circle is covered by the rotation.

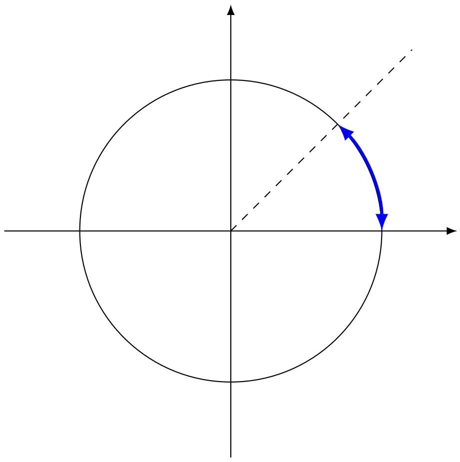
Up till now, the measure for angles was done in degrees. One degree (1°) is the angle defined by $\frac{1}{360}$th of a circle i.e., one complete turn is 360°.

⚠ This measure does not depend on the size of the circle.

Radian measure

Another way to measure an angle is to consider the proportion of a complete circle by comparing the arclength determined by the angle with the arclength of the complete circle.

For instance one eighth of a circle (half of a right angle):



The complete circle (circumference) measure $\pi \cdot 2 \cdot r$ where r is the radius.
The blue arc is one eighth of the circumference: $\frac{2\pi r}{8} = \frac{\pi}{4} \cdot r$.
This measure depends on the radius, so we define:

Definition 23
Consider a circle centred on the vertex of an angle, and the arclength determined by the intersection of the arms of the angle with the circumference of the circle.

The angle in radian measure is the ratio $\frac{\text{arclentgth}}{\text{radius}}$

Thus the angle above becomes simply $\frac{\pi}{4}$.
Notice that the length units have cancelled!

Exercise 86
In radian measure:
Considering a circle of radius r:

(1) What is the measure of a complete turn (a round angle)?

(2) What is the measure of a right angle (a quarter turn)?

(3) What is the measure of a flat angle (half a turn)?

Exercise 87
Find the formula for transforming a degree measure to a radian measure.

Exercise 88
Transform the following into radian measure:

(1) $60°$

(2) $22.5°$

(3) $30°$

(4) $15°$

(5) $7.5°$

(6) $120°$

(7) $10°$

(8) $1°$

(9) $452°$

Exercise 89

Find the formula for transforming a radian measure to a degree measure.

Exercise 90

Transform the following into degree measure:

(1) 1

(2) $\frac{\pi}{3}$

(3) 0.1

(4) $\frac{2\pi}{3}$

(5) $\frac{7\pi}{4}$

(6) 15π

(7) 16π

(8) 3

(9) 1.5

Exercise 91

One nautical mile is the length of arc of 1 minute at the surface of the Earth. (One minute is the sixtieth of a degree, the sixtieth of a minute is a second.)

What is one minute of arc in radian measure?

What is the length of one nautical mile? [1]

Exercise 92

A circular sector is the part of a disc lying between two radii.

Find the area of a sector of angle θ (in radian measure).

Exercise 93

A circle has a radius of 2.5m. Find the area of a sector of angle $\frac{3\pi}{4}$

Exercise 94

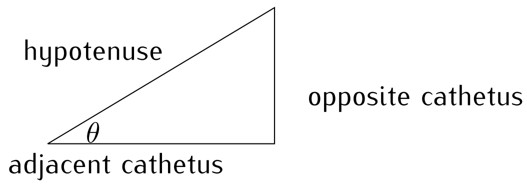
In a unit circle, what is the angle of a sector of area 1? (radian measure)

⚠ In analysis, only radian measure is used.

[1] Because the Earth is not a perfect sphere, the official length of the nautical mile is now slightly different and does not depend on which part of the Earth you are.

Trigonometric ratios

In a right-angled triangle the following ratios have been defined:



$$\sin(\alpha) = \frac{\text{opposite cathetus}}{\text{hypotenuse}}$$

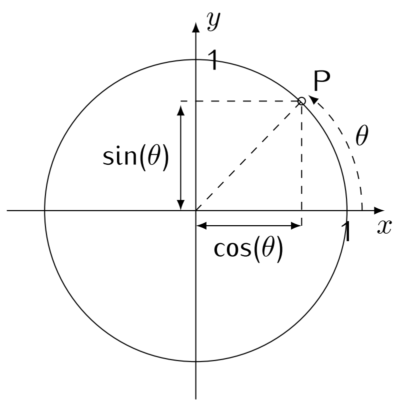
$$\cos(\alpha) = \frac{\text{adjacent cathetus}}{\text{hypotenuse}}$$

$$\tan(\alpha) = \frac{\text{opposite cathetus}}{\text{adjacent cathetus}}$$

Circular functions

We now redefine these ratios in the **trigonometric circle** which has a radius equal to 1 centred on the origin and angle direction is anti-clockwise, starting from the positive abscissa semiline.

The original definitions work only for positive values (lengths are always positive) i.e., only for the first quadrant. These definitions are now **extended** to the whole unit circle: the sine is the vertical coordinate of a point, the cosine is its horizontal coordinate, the position of the point given by its arc-length determines the angle



As the radius is 1, we then have

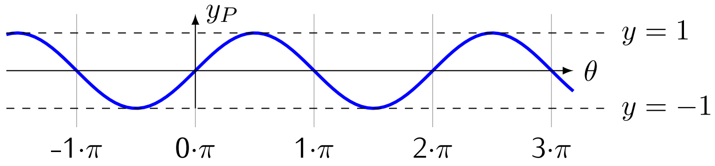
$$\sin(\theta) = \frac{\text{opposite cathetus}}{1} = \text{opposite cathetus} = y_P$$

$$\cos(\theta) = \frac{\text{adjacent cathetus}}{1} = \text{adjacent cathetus} = x_P$$

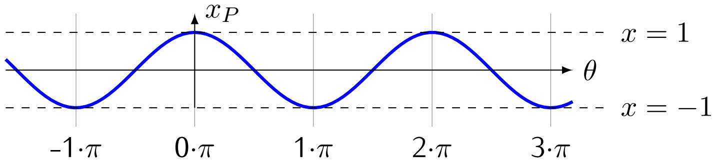
$$\tan(\theta) = \frac{\text{opposite cathetus}}{\text{adjacent cathetus}} = \frac{\sin(\alpha)}{\cos(\alpha)}$$

If we unroll the circle, the values of $\sin$, $\cos$ and $\tan$ as functions of θ appear as in the following:

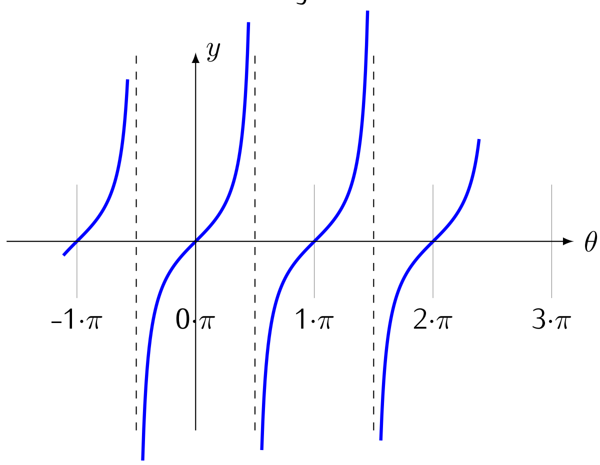
The Sine curve



The Cosine curve



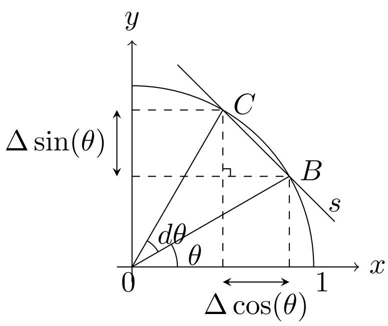
The Tangent curve



Properties of circular functions

Property 27	Sine and cosine are continuous functions.

Consider the trigonometric circle. The chord BC is shorter than the arc BC.

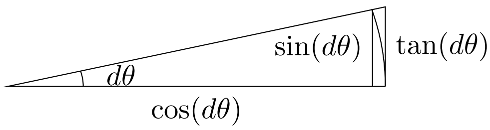


Proof
By Pythagoras: $(\Delta\sin(\theta))^2 + (\Delta\cos(\theta))^2 = (BC)^2 < (d\theta)^2 \simeq 0$
This implies both $\Delta\sin(\theta) \simeq 0$ and $\Delta\cos(\theta) \simeq 0$ $\square$

Property 28

$$\frac{\sin(d\theta)}{d\theta} \simeq 1$$

Proof
We consider $d\theta > 0$



Comparing the area of the sector with that of the inside and outside triangles, we obtain

inside triangle $\leq$ sector $\leq$ outside triangle.

Inside triangle area: $\frac{\cos(d\theta)\cdot\sin(d\theta)}{2}$
Sector area : $\frac{d\theta}{2}$ (Note that this result is not valid for angles in degrees.)
Outside triangle: $\frac{1\cdot\tan(d\theta)}{2} = \frac{\frac{\sin(d\theta)}{\cos(d\theta)}}{2} = \frac{\sin(d\theta)}{2\cos(d\theta)}$
Dividing everything by $\sin(d\theta)/2$ (which is positive) we obtain

$$\cos(d\theta) < \frac{d\theta}{\sin(d\theta)} < \frac{1}{\cos(d\theta)}$$

By continuity, $\cos(d\theta) \simeq 1$. So $\frac{1}{\cos(d\theta)} \simeq 1$ also, hence $\frac{d\theta}{\sin(d\theta)} \simeq 1$.

By symmetry through the x-axis, the same holds for $d\theta < 0$ $\square$

Property 29

$$\frac{1-\cos(d\theta)}{d\theta} \simeq 0$$

Proof
Multiply above and below by $(1+\cos(d\theta))$
This yields

$$\frac{1-\cos^2(d\theta)}{d\theta\cdot(1+\underbrace{\cos(d\theta)}_{\simeq 1})} \simeq \frac{\sin^2(d\theta)}{2\cdot d\theta} = \underbrace{\frac{\sin(d\theta)}{d\theta}}_{\simeq 1}\cdot\underbrace{\sin(d\theta)}_{\simeq 0}\cdot 1/2 \simeq 0$$

$\square$

Property 30

(1)
$$\sin'(\theta) = \cos(\theta)$$

(2)
$$\cos'(\theta) = -\sin(\theta)$$

Proof

We use
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
to expand $\Delta \sin(\theta) = \sin(\theta + d\theta) - \sin(\theta)$, we have:

$$\begin{aligned}
&\Delta \sin(\theta) \\
&\qquad = \sin(\theta)\cos(d\theta) + \cos(\theta)\sin(d\theta) - \sin(\theta) \\
&\qquad = \sin(\theta) \cdot (\cos(d\theta) - 1) + \cos(\theta)\sin(d\theta)
\end{aligned}$$

divide by $d\theta$

$$\frac{\Delta \sin(\theta)}{d\theta} = \sin(\theta) \cdot \underbrace{\frac{(\cos(d\theta) - 1)}{d\theta}}_{\simeq 0} + \cos(\theta) \underbrace{\frac{\sin(d\theta)}{d\theta}}_{\simeq 1} \simeq \cos(\theta)$$

hence $\sin'(\theta) = \cos(\theta)$. □

Exercise 95

Use
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
to prove that $\cos'(\theta) = \sin(\theta)$.

Exercise 96

Use $\tan(x) = \frac{\sin(x)}{\cos(x)}$ to prove that $\tan'(x) = 1 + \tan^2(x)$

The sine function assigns a (vertical) coordinate to an angle. The inverse function of the sine function is the function that assigns an angle to a vertical coordinate. If $\sin(\theta) = y$, then the inverse function of sine is the arc whose vertical coordinate is y, hence the name **arcsine**. Symbol: $\arcsin(y) = \theta$, also noted on calculators as $\sin^{-1}(y) = \theta$, or, depending on the brand: $\text{asin}(y) = \theta$.

⚠ $\sin^{-1}$ is not $\frac{1}{\sin}$.

Property 31

(1)
$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

(2)
$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$$

(3)
$$\arctan'(x) = \frac{1}{1+x^2}$$

Proof

These are inverse functions. We use property 20, page 61, and the $f(x) = y$ notation.

(1)
$$\sin(y) = x \text{ and } \frac{dx}{dy} = \cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$$

hence
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

(2)
$$\cos(y) = x \text{ and } \frac{dx}{dy} = -\sin(y) = \sqrt{1-\cos^2(y)} = \sqrt{1-x^2}$$

hence
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

(3)
$$\tan(y) = x \text{ and } \frac{dx}{dy} = 1+\tan^2(y) = \sqrt{1+x^2}$$

hence
$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$\square$

Exercise 97

Compute the derivatives of the following:

(1) $f : x \mapsto \sin^2(3x+\pi)$

(2) $g : x \mapsto x \cdot \sin(x^2+1)$

(3) $h : x \mapsto \sin^2\left(\frac{x}{x^2+1}\right) + \cos^2\left(\frac{x}{x^2+1}\right)$

(4) $j : x \mapsto 1 + \tan^2(x)$

Exercise 98

(1) Show that $f : x \mapsto \sin^6(x) + \cos^6(x) + 3\sin^2(x)\cos^2(x)$ is a constant function. (Hint: use the derivative...)

(2) At what values does $f : x \mapsto \sin(x) + \cos(x)$ have stationary points?

(3) What is the equation of the straight line tangent to $y = \sin^2(x)$ at $x = \frac{\pi}{4}$?

Example: Consider the integral
$$\int_0^{\pi/2} x \cdot \sin(x) \cdot dx$$
To integrate by parts, use $f' : x \mapsto \sin(x)$ et $g : x \mapsto x$. We have $f(x) = -\cos(x)$ and $g'(x) = 1$ hence
$$\int_0^{\pi/2} x \cdot \sin(x) \cdot dx = -x \cdot \cos(x)\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos(x) \cdot dx = \sin(x)\Big|_0^{\pi/2} = 1$$
We also deduce that
$$\int x \cdot \sin(x) \cdot dx = -x \cdot \cos(x) + \sin(x) + C$$

Exercise 99

Use integration by parts to compute the following integrals:

(1) $\displaystyle\int x \cdot \cos(x) \cdot dx$

(2) $\displaystyle\int (\cos(x))^2 \cdot dx$

(3) $\displaystyle\int x^2 \cdot \sin(x) \cdot dx$

(4) $\displaystyle\int \sin(x) \cdot \cos(x) \cdot dx$

Exercise 100

Compute the following integrals:

(1) $\displaystyle\int 2x \cdot \sin(x^2) \cdot dx$

(2) $\displaystyle\int x^2 \cdot (x^3 + 1) \cdot dx$

(3) $\displaystyle\int \sin(x) \cdot \cos(\cos(x)) \cdot dx$

(4) $\displaystyle\int \sin(x) \cdot \cos^2(x) \cdot dx$

Summary of this chapter

- $\sin'(x) = \cos(x)$
- $\cos'(x) = \sin(x)$
- $\tan'(x) = \dfrac{1}{\cos^2(x)}$
- also $\tan'(x) = 1 + \tan^2(x)$
- $\arcsin'(x) = \dfrac{1}{\sqrt{1 - x^2}}$
- $\arccos'(x) = -\dfrac{1}{\sqrt{1 - x^2}}$
- $\arctan'(x) = \dfrac{1}{1 + x^2}$

Chapter 11

Transcendental functions

A transcendental function is a function that cannot be written as a combination of polynomials, rational functions, root functions or compositions of these.

Given this definition, all circular functions are transcendental functions, together with the ones that we will introduce in this chapter.

The functions studied here are of extreme importance in all areas of science and engineering, and although they are linked to trigonometry, they deserve a specific chapter.

Antiderivative of $x \mapsto \frac{1}{x}$

Let n be a positive integer. From $(x^{n+1})' = (n+1) \cdot x^n$ we can deduce

$$\int x^n \cdot dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$$

Hence an antiderivative of $x \mapsto \dfrac{1}{x}$ is **not** a particular case of this formula.

Though $\frac{1}{x}$ is not transcendental, it turns out that its antiderivative is quite special!

Exercise 101

Let f be an antiderivative of $x \mapsto \frac{1}{x}$ (why is there one?) Then f is strictly increasing (why?) and so it has an inverse, call it g.

Show that this implies $g'(x) = g(x)$.

Practice exercise PE25 Answer page 129

Let $a, b > 0$. Considering an antiderivative of $\frac{1}{t}$, use the substitution $u = \frac{t}{a}$ to show that

$$\int_a^{a \cdot b} \frac{1}{t} \cdot dt = \int_1^b \frac{1}{u} \cdot du$$

We now consider an antiderivative given by

$$f(x) = \int_1^x \frac{1}{t} \cdot dt = \int_1^x \frac{1}{u} \cdot du \qquad (*)$$

Then

$$f(a)+f(b)=\int_1^a \frac{1}{u}\cdot du+\int_1^b \frac{1}{u}\cdot du=\int_1^a \frac{1}{u}\cdot du+\int_a^{ab} \frac{1}{u}\cdot du=\int_1^{ab} \frac{1}{u}\cdot du$$

From this result, we deduce that $f(a)+f(b)=f(a\cdot b)$:

Exercise 102

Let $a>0$ and b a rational number. Considering f to be an antiderivative of $\frac{1}{x}$ as (*) page 95, show that

$$f(a^b)=b\cdot f(a)$$

(To find the substitution, consider the transformation of the bounds.)

Exercise 103

What kind of function has the properties $f(a\cdot b)=f(a)+f(b)$ and $f(a^b)=b\cdot f(a)$?

Property 32 The antiderivative f of $\frac{1}{x}$ satisfies the following limits:

$$\lim_{x\to 0^+} f(x)=-\infty \quad \text{and} \quad \lim_{x\to+\infty} f(x)=+\infty$$

Exercise 104

Prove property 32. Hint: for ultralarge x use ultralarge N such that $2^N\le x$.

Definition 24 The **natural logarithm** is the function $\ln :]0,+\infty[\to \mathbb{R}$ defined by

$$x\mapsto \int_1^x \frac{1}{t}\cdot dt$$

To indicate the natural logarithm function, we use the notation $ln(x)$.

Definition 25 e is the unique number such that

$$\ln(e)=1$$

e is an irrational number whose first digits are

$$e=2.71828\ldots$$

Definition 26 The **exponential** function $\exp : \mathbb{R} \longrightarrow]0, +\infty[$ is the inverse of $\ln$

Thus $\ln$ is in fact $\log_e$ and $\ln(e) = 1$

We have, for rational x, that $a^x = \exp(x \ln(a))$, hence $e^x = \exp(x)$. For irrational x, we **define** a^x to be $\exp(x \ln(a))$ hence also $e^x = \exp(x)$ for all x.

We also have $\ln(a^y) = y \cdot \ln(a)$ for all y. Writing $x = a^y$ we get $\ln(x) = \log_a(x) \cdot \ln(a)$ so $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

Property 33

(1) Let $b \in \mathbb{R}$. The function $x \mapsto x^b$ is differentiable on its domain and $(x^b)' = b \cdot x^{b-1}$, for all $x \in \mathbb{R}$.

(2) Let $a > 0$. The base a exponential is differentiable on its domain and $(a^x)' = \ln(a) \cdot a^x$, for $x > 0$

(3) Let $a > 0$. The base a logarithm is differentiable and $(\log_a(x))' = \frac{1}{\ln(a) \cdot x}$

Exercise 105

Prove property 33.

Exercise 106

Let f be a positive real function whose derivative is continuous. Calculate:

$$\int \frac{f'(x)}{f(x)} \cdot dx$$

Exercise 107

Calculate

$$\int \tan(x) \cdot dx$$

Exercise 108

Let f be a positive real function whose derivative is continuous. Calculate:

$$\int f'(x) \cdot e^{f(x)} \cdot dx$$

Exercise 109

Using $\ln(x) = 1 \cdot \ln(x)$, use integration by parts to compute $\int \ln(x) dx$

Exercise 110

(1) Differentiate $\ln(x)$

(2) Differentiate e^x

(3) Integrate $x \mapsto e^x$

(4) Differentiate the function $x \mapsto \ln(\ln(x))$

(5) Differentiate the function $x \mapsto \ln(x^a)$ (Note that a is not the variable!)

(6) Differentiate the function $x \mapsto \ln(a^x)$

(7) Differentiate $x \mapsto e^{x^2}$

(8) Using the fact that $u = e^{ln(u)}$ (if $u > 0$) differentiate $x \mapsto a^x$ (for $a > 0$ and $x > 0$)

(9) Same idea: Differentiate the function $x \mapsto x^x$

Exercise 111

Differentiate $\ln(|x|)$

This proves the following extension:

Property 34 The antiderivative of $\frac{1}{x}$ is $\ln(|x|) + K$ for $K \in \mathbb{R}$.

Summary of this chapter

- $\ln'(x) = \dfrac{1}{x}$
- $(e^x)' = e^x$

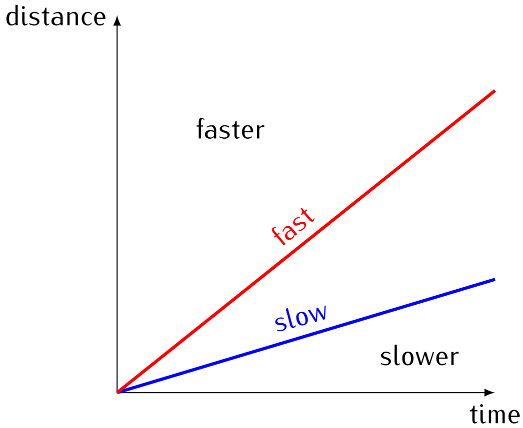
Chapter 12

Curve sketching – advanced functions and techniques

Bending

If the velocity of an object is constant over time, the graph with time as independent variable (horizontal axis) and distance as dependent variable (vertical axis) is a straight line.

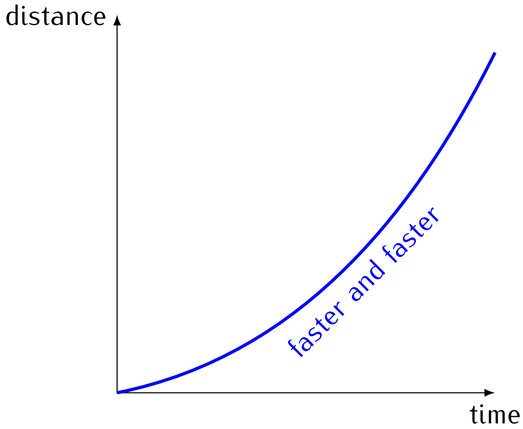
The faster the object, the steeper the line



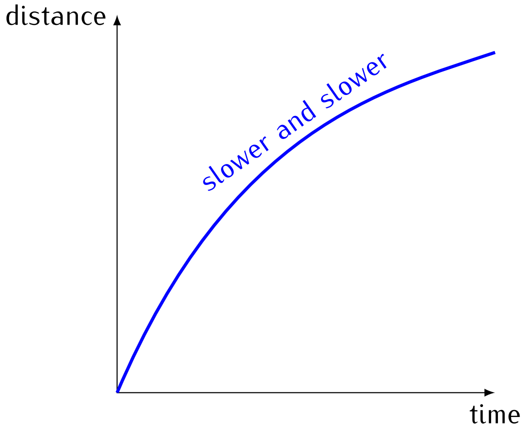
The velocity is given by $\frac{\Delta f(a)}{\Delta t}$ (unit: distance over time i.e., $\frac{m}{s}$) and it is well known that for straight lines, the slope is the same whichever increment is chosen for Δt.

For an object moving at a varying velocity, the velocity at a given instant a is given by the derivative at a.

If an object moves faster and faster, its position graph will **bend upwards**: it is accelerating.



If an object moves slower and slower, its position graph will **bend downwards**: it is decelerating.



The variation of speed over time is an acceleration (or a deceleration).

The variation of the derivative of the position over time is the variation of the variation of the position over time: it is the derivative of the derivative.

The derivative of the derivative of a function f is its **second derivative** and is symbolised by f''.

Definition 27 Let f be differentiable on an open interval and f' be its derivative function. If f' is also differentiable, then $f''(x) = (f')'(x)$ and is called the second derivative of f at x.

If f' is differentiable, we say that f is twice differentiable.

Property 35 The second order increment equation for f twice differentiable at a, is

$$f(a + dx) = f(a) + f'(a) \cdot dx + \frac{f''(a)}{2} \cdot dx^2 + \varepsilon \cdot dx^2$$

with $\epsilon \simeq 0$.

Proof

For the proof of this property, we look for a second degree polynomial $q(x)$ which approximates $f(x)$ in the following way

$$x \simeq a \Rightarrow f(x) - q(x) = \varepsilon \cdot (x-a)^2 \simeq 0$$

The increment equation approximates the function with an error in $\varepsilon \cdot dx^2$, which can also be written — for $x \simeq a$ — $\varepsilon \cdot (x-a)^2$ because in the following proof we consider x as a variable, so we write $(x-a)$ instead of dx.

We write $q(x) = b_0 + b_1(x-a) + b_2(x-a)^2$ hence

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \varepsilon \cdot (x-a)^2$$

For $x = a$ we get $f(a) = b_0 + 0$, hence

$$q(x) = f(a) + b_1(x-a) + b_2(x-a)^2$$

and

$$f(x) = f(a) + b_1(x-a) + b_2(x-a)^2 + \varepsilon \cdot (x-a)^2 \tag{*}$$

Now we write $\delta = (b_2 + \varepsilon)(x-a) \simeq 0$ (recall that $x \simeq a$) and (*) becomes

$$f(x) = f(a) + b_1 \cdot (x-a) + \delta \cdot (x-a)$$

which by the increment equation (see page 44) shows that $b_1 = f'(a)$

So

$$f(x) = f(a) + f'(a)(x-a) + b_2(x-a)^2 + \varepsilon \cdot (x-a)^2$$

We differentiate this expression (with respect to x, hence $f(a)$ and $f'(a)$ are constants.

$$f'(x) = f'(a) + 2 \cdot b_2 \cdot (x-a) + 2 \cdot \varepsilon \cdot (x-a) \tag{**}$$

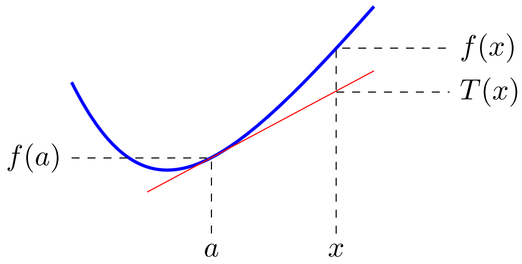
Rewriting the increment equation for $f'(x) = f'(a) + f''(a) \cdot dx + \varepsilon \cdot dx$ and comparing with (**) we see that $2 \cdot b_2 = f''(a)$ hence that $b_2 = \dfrac{f''(a)}{2}$. □

Definition 28 Let f be differentiable on a.
The curve of f is **bending upwards at** a if $f(x)$ is above the line tangent to f at $\langle a, f(a) \rangle$, i.e.,

$$f(x) \geq f(a) + f'(a)(x-a) \text{ whenever } x \simeq a$$

The curve of f is **bending downwards at** a if $f(x)$ is above the line tangent to f at $\langle a, f(a) \rangle$, i.e.,

$$f(x) \leq f(a) + f'(a)(x-a) \text{ whenever } x \simeq a$$



Property 36 (Bending and second derivative) Let f be twice differentiable at a. Then

(1) If $f''(a) > 0$ then f is bending upwards at a.

(2) If $f''(a) < 0$ then f is bending downwards at a.

Proof
Using the second order increment equation (see page 100) we write

$$f(a + dx) = f(a) + f'(a) \cdot dx + \frac{1}{2} f''(a) \cdot dx^2 + \varepsilon \cdot dx^2$$

since $f''(a) > 0$ and is observable, $f''(x) + \varepsilon > 0$ so $f(a + dx) > f(a) + f'(a) \cdot dx$. The proof is similar for $f''(a) < 0$.
$\square$

Naturally, if f is bending upwards, respectively downwards, at every point of an open interval I, the function is bending upwards, respectively downwards on I.

If $f''(a) = 0$ then f is not bending at a.

Another consequence is that if a function f has a maximum at an interior point a of an interval, then (assuming $f''(a)$ exists) $f''(a) \leq 0$ and if it is a minimum, then $f''(a) \geq 0$.

Definition 29 (Inflexion point) The point where $f''(x) = 0$ is called an **inflexion point.**
If $f''(c) = 0$ then $\langle c, f(c) \rangle$ is an inflexion point.

See example on page 105 for the use of finding inflexion points when drawing a curve.

Rule of de L'Hospital

Theorem 8 (Rule of de L'Hospital for $0/0$)
Let f and g be differentiable functions at a. Suppose that $f(a) = g(a) = 0$, but that $g'(a) \neq 0$. Then

$$\frac{f(a + dx)}{g(a + dx)} \simeq \frac{f'(a)}{g'(a)}$$

Proof

By the increment equation (page 44)

$$\frac{f(a+dx)}{g(a+dx)} = \frac{f'(a)\cdot dx + \epsilon \cdot dx}{g'(a)\cdot dx + \delta \cdot dx}$$

We use ε and δ since the ultrasmall quantity in the increment equation is not necessarily the same for both functions.

Upon dividing top and bottom by dx and remembering that $\epsilon \simeq 0$ and $\delta \simeq 0$, we have:

$$\frac{f(a+dx)}{g(a+dx)} = \frac{\frac{f(a+dx)}{dx}}{\frac{g(a+dx)}{dx}} = \frac{f'(a)+\epsilon}{g'(a)+\delta} \simeq \frac{f'(a)}{g'(a)}$$

□

When looking for horizontal asymptotes, we use ultralarge values of x. This can lead to situations such as $\frac{f(x)}{g(x)}$ with $f(x) \simeq 0 \simeq g(x)$. This is not exactly the situation of theorem 8 which uses in the proof that the quotient is between two values exactly zero.

Restated here as:

If $f(x) \simeq 0$ and $g(x) \simeq 0$ for all ultralarge $x > 0$, then $\frac{f(x)}{g(x)} \simeq \frac{f'(x)}{g'(x)}$ for all ultralarge $x > 0$

Nonetheless, the theorem does hold also in this case, but the proof is beyond the scope of this course.

The rule of de L'Hospital also holds in the case of a quotient between two ultralarge values. Again, a complete proof of this case is beyond the scope of this course. What we will prove here is that if it holds for the case $\frac{\text{ultrasmall}}{\text{ultrasmall}}$, then it also holds for the case $\frac{\text{ultralarge}}{\text{ultralarge}}$

Assume that for some x in the domain, $f(x)$ and $g(x)$ are ultralarge for all $x \simeq a$. Then $\frac{f(x)}{g(x)} = \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$ which is $\frac{\text{ultrasmall}}{\text{ultrasmall}}$ so

$$\frac{f(x)}{g(x)} \simeq \frac{\left(\frac{1}{g(x)}\right)'}{\left(\frac{1}{f(x)}\right)'} = \frac{-\frac{g'(x)}{g^2(x)}}{-\frac{f'(x)}{f^2(x)}} = \frac{g'(x)}{f'(x)}\frac{f^2(x)}{g^2(x)}$$

Hence

$$\frac{f(x)}{g(x)} \simeq \frac{g'(x)}{f'(x)}\frac{f^2(x)}{g^2(x)}$$

which leads to

$$\frac{g(x)}{f(x)} \simeq \frac{g'(x)}{f'(x)}$$

which can be transformed into

$$\frac{f(x)}{g(x)} \simeq \frac{f'(x)}{g'(x)}$$

which is what we wanted to show.

Practice exercise PE26 Answer page 129

Evaluate using de L'Hospital's rule.

(1) $\dfrac{\sqrt{9+x}-3}{x}$ for $x \simeq 0$

(2) $\dfrac{2-\sqrt{x+2}}{4-x^2}$ for $x \simeq 2$

(3) $\dfrac{\sqrt{u+1}+\sqrt{u-1}}{u}$ for ultralarge u

(4) $\dfrac{(1-x)^{1/4}-1}{x}$ for $x \simeq 0$

(5) $\left(\dfrac{1}{t}+\dfrac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$ for $x \simeq 0_+$

(6) $\dfrac{(u-1)^3}{u^{-1}-u^2+3u-3}$ for $u \simeq 1$

(7) $\dfrac{1+5/\sqrt{u}}{2+1/\sqrt{u}}$ for $u \simeq 0_+$

(8) $\dfrac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$ for ultralarge x

(9) $\dfrac{1-t/(t-1)}{1-\sqrt{t/(t-1)}}$ for ultralarge t

Exercise 112

Evaluate using de L'Hospital's rule.

(1) $\dfrac{1/t-1}{t^2-2t+1}$ for $t \simeq 1$ (with $(t>1)$)

(2) $\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ for $x \simeq 1$

(3) $\dfrac{x^2}{\sqrt{2x+1}-1}$ for $x \simeq 0$

(4) $\dfrac{2+1/t}{3-2/t}$ for $t \simeq 0$

(5) $\dfrac{x+5-2x^{-1}-x^{-3}}{3x+12-x^{-2}}$ for ultralarge x

(6) $\left(t+\dfrac{1}{t}\right)((4-t)^{3/2}-8)$ for $t \simeq 0$

(7) $\dfrac{u+u^{-1}}{1+\sqrt{1-u}}$ for ultralarge u

Curve sketching

To sketch the curves of the functions proposed in this chapter, the rule of de L'Hospital may be also required. The functions may include any combination of functions studied up to now. Some functions may be difficult.

To determine the points of maximum and minimum value of the functions, instead of analysing the change in sign of the first derivative of the function, we will check if the second derivative of the function is positive or negative at the critical point. Also, the computation of the second derivative will allow us to find any inflexion points.

The steps to be taken are now the following:

(1) Find the domain.

(2) Find the roots and the intercepts (if any).

(3) Find the asymptotes (if any).

(4) Find the first derivative (if any).

(5) Find the roots of the first derivative (if any).

(6) Find the second derivative (if any).

(7) Determine the maximum and minimum values.

(8) Find the roots of the second derivative (if any), the inflexion points and bending direction.

(9) Use this information to choose a convenient scale.

(10) Sketch the function.

Note that finding extrema can be done using the second derivative (negative at a maximum, positive at a minimum) or by checking whether there is a change of the signum of the first derivative (from positive to negative at a maximum, from negative to positive at a minimum).

Example: As in the example page 63, we use

$$f : x \mapsto \frac{x}{x^2+1}$$

and recollect the results up to point (5):

	$-\infty$		-1		0		1		∞
f	$\simeq 0$	$-$	$-$	$-$	0	$+$	$+$	$+$	$\simeq 0$
f'			0				0		

(6)

$$f''(x) = \frac{-2x(x^2+1)^2 - (1-x^2)2(x^2+1)2x}{(x^2+1)^4} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

(7) From the expression for $f''(x)$ we calculate: $f''(-1) = \frac{1}{2} > 0$ and $f''(1) = -\frac{1}{2} < 0$; hence, there is a minimum at $x = -1$ and a maximum at $x = 1$ (compare the results with those at page 64).

(8) $f''(x) = 0 \Rightarrow x \in \{-\sqrt{3}, 0, \sqrt{3}\}$. Bending changes at $\pm\sqrt{3}$ and 0.

	$-\infty$		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		∞
f	$\simeq 0$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$\simeq 0$
f'					0				0				
f''		$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	
f		$\searrow$	$\searrow$	$\searrow$	min	$\nearrow$	$\nearrow$	$\nearrow$	max	$\searrow$	$\searrow$	$\searrow$	
		$\cap$	$infl$	$\cup$	$\cup$	$\cup$	$infl$	$\cap$	$\cap$	$\cap$	$infl$	$\cup$	

(9) A convenient scale can be chosen by considering:

$f(-1) = -\frac{1}{2}$

$f(1) = \frac{1}{2}$

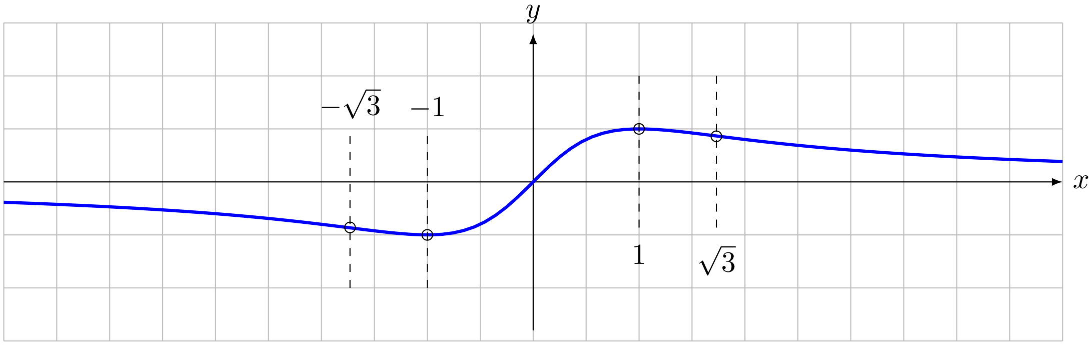
$f(-\sqrt{3}) = -\frac{\sqrt{3}}{4} \approx -0.43$

$f(\sqrt{3}) = \frac{\sqrt{3}}{4} \approx 0.43$

and by taking $\sqrt{3} \approx 1.73$.[1]

[1]Notice that $\sqrt{3} \approx 1.73$, but $\sqrt{3} \not\simeq 1.73$.

(10)



Practice exercise PE27 Answer page 129
Sketch the curves of the following functions:

- $f : x \mapsto x\ln(x)$
- $g : x \mapsto \dfrac{x}{\ln(x)}$
- $h : x \mapsto \dfrac{e^x}{1+e^x}$
- $j : x \mapsto \dfrac{1}{1+e^x}$
- $k : x \mapsto \ln(x^2+1)$
- $l : x \mapsto e^{-x^2}$

Summary of this chapter

The second derivative gives information about the bending of a function and introduces a new kind of critical point: the inflexion point (a place where, locally, there is no bending).

Analysis deals extensively with fractions that are of the form $\frac{0}{0}$ or very close to these undefined fractions. The theorem of de l'Hospital produce an extra method for determining such values when both denominator and numerator are differentiable functions.

Chapter 13

Applications

In the following exercises, derivatives and antiderivatives will be used. While derivatives are reasonably straightforward to establish, antiderivatives need a bit a backward thinking.

Exercise 113

Find the antiderivatives of the following functions:
Caution: sometimes the variable is x, sometimes t!

(1) $f(x) = e^x$

(2) $g(t) = e^{a \cdot t}$

(3) $h(t) = b \cdot e^{a \cdot t}$

(4) $j(x) = \sin(a \cdot x)$

(5) $k(t) = b \cdot \cos(a \cdot t)$

Exercise 114

Sometimes a relation between a function and its derivative is given – the function being unknown. These are **differential equations**. They are beyond the scope of this course in their general form, yet some cases can be solved.

Find $f(x)$ (or y when noted) for the following: (Do not forget that if $f' = g'$, then $f = g + K$ for $K \in \mathbb{R}$)

(1) $f'(x) = f(x)$

(2) $y' = a \cdot y$

(3) $f'(x) = -f(x)$

(4) $f'(x) = -a \cdot f(x)$

(5) (with a second derivative!) $y'' = -y$

Population growth with unlimited resources

Suppose you want to find a mathematical formula describing the growth of the population of a given species (humans, livestock, bacteria, ...). Here, for vocabulary reasons, we consider humans.

We build a model using a certain number of assumptions (they could result in an over-simplification of the problem; in which case, we use the model itself to find out more relevant assumptions and rewrite the model). The assumptions we make are:

(1) The average number of children per human being is constant.

(2) The average life span is constant.

(3) The variation in population (i.e., births minus deaths) over a given amount of time is proportional to the population itself (after all, if the population doubles, so do the births and deaths).

We write $p(t)$ for the population depending on time (independent variable is noted t not x) and we denote the proportion mentioned above by k. It is the individual growth rate, i.e. the per capita change in population over time; a possible unit of measure could be new individuals per year per capita.

Then we have

$$p'(t) = k \cdot p(t) \tag{13.1}$$

This is a **differential equation**.

We know that if $k = 1$ we have

$$p'(t) = p(t) \tag{13.2}$$

then $p(t) = e^t = p'(t)$ satisfies this property.

Exercise 115

Show that for any real number C, the function $p(t) = C \cdot e^t$ also satisfies (13.2)

Exercise 116

Show that $C \cdot e^{kt}$ is also a solution to (13.1)

Practice exercise PE28 Answer page 130

In 2020, the world population was estimated to be 7.8 billion and the annual growth rate is 1.05%.

Consider time t is in years and $t = 0$ is year 2020.

(1) Find the formula the expresses the world population with respect to time.

(2) How long will it take to double the world population?

This solution to the population growth problem has a major flaw: if the population doubles every 66 years, in 660 years it will have doubled 10 times, which is a $2^{10} = 1024$ factor. This would yield a population of around 8 hundred billion people. It seems highly improbable that the Earth could support such a population. This model can only be used as an approximation when the ecosystem is far from saturated.

As a first conclusion, the assumption that the growth rate is constant must be changed. One possibility is that he growth rate depends on how big the population is.

Rabbits on an island

Considering a model less dramatic than human population, we look at rabbits on an island. At first they have enough to eat and reproduce freely.

But at some point in time, food becomes harder to find, there is not enough space to dig new burrows and the growth rate will decrease, until eventually an equilibrium is reached.

This model is mathematically much more complicated than the previous one. It will not be possible to show how the solution is found. The solution will be given without proof but all the other mathematics involved should be understood.

Assume the growth rate per individual depends on the total population size, hence

$$p'(t) = p(t) \cdot f(p(t)) \tag{13.3}$$

where $f(p(t))$ is the individual growth rate.

Writing $p(t) = y$ we have

$$y' = y \cdot f(y)$$

When the population is high, it has inhibitory effects on the growth rate hence we assume $\frac{df(y)}{dy}$ to be negative.

Note that we wrote $\frac{df(y)}{dy}$ as in the chain rule: the variation of f with respect to y not t.

The simplest assumption is to choose a linear function:

$$f(y) = a - by \qquad a, b > 0$$

hence formula (13.3) becomes[1]

$$y' = y(a - by)$$

or

$$p'(t) = p(t) \cdot (a - b \cdot p(t))$$

When $p(t)$ is small and assuming $b < 1$, we have $p'(t) \approx p(t) \cdot a$ where a is the growth rate if no inhibiting conditions existed.

To understand the b coefficient, we rewrite the equation as

[1] Pierre-François Verhulst (in 1838)

$$p'(t) = a \cdot p(t) \left(\frac{\frac{a}{b} - p(t)}{\frac{a}{b}} \right)$$

We observe that $p'(t) = 0$ if $p(t) = 0$ and also if $p(t) = a/b$. Then a/b would be the maximum sustainable population and $\frac{\frac{a}{b} - p(t)}{\frac{a}{b}}$ is the proportion of that maximum population which has not been realised yet.

Rewriting once more with $a/b = N$:

$$p'(t) = a \cdot p(t) \cdot \left(1 - \frac{p(t)}{N}\right) \tag{13.4}$$

Exercise 117

Check that

$$p(t) = \frac{N}{1 + \left(\frac{N-C}{C}\right) e^{-at}}$$

is a solution to (13.4).

As for the case of population growth with unlimited resources, C is the initial population.

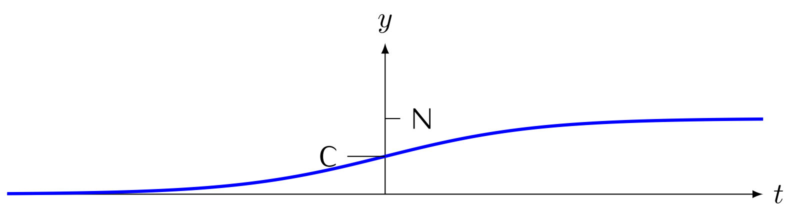
Exercise 118

Calculate the following limits (for p as in exercise above):

(1) $\lim_{t \to \infty} p(t)$

(2) $\lim_{t \to -\infty} p(t)$

In this last case, the limit to $-\infty$ would be when the population started.



This curve is called the logistics curve.

It has a major drawback: in order to predict the population growth, one needs to know the maximal possible population... but it is a better model than the straightforward exponential.

Epidemic spread

Assume that there is an epidemic of a mild (non-lethal) disease. The epidemic does not last too long so the natural death rates and birth rates can be ignored hence the total population remains constant.

The population is divided into three categories:

(1) Susceptible (S), those who have not (yet) been ill

(2) Infected (I)

(3) Removed (R) – or Recovered – those who having been infected have cured and are immune so they do not become sick again.

It is assumed that at the start, nobody is immune, hence $R(0) = 0$.

(It is possible to consider any proportion of the original population to be immune and observe the overall effect of, say, a vaccination.)

This is called the SIR model, and it leads to differential equations impossible to solve with this course. A very simple model is used here.

Exercise 119

The initial population is 10 000 people; we take a unit to be one thousand, hence the population is 10.

The number of people who become sick is big at first and decreases to end after 15 weeks by everybody having caught the disease, hence (for t in weeks) we have

$S(0) = 10$, meaning that for $t = 0$ nobody is infected nor immune, hence everybody is susceptible to get ill.

$S(15) = 0$

The function is a quadratic. Its only zero is at $t = 15$.

Find the equation of S.

Sketch the function S.

Exercise 120

The disease lasts one week, hence after one week, people start recovering, until the sixteenth week, when all have recovered. This implies that $R(0) = 0$ and $R(16) = 10$.

Moreover, it is assumed that the initial recovery rate is very small ($R'(0) = 0$) and the final recovery rate also satisfies $R'(16) = 0$.

Find the polynomial of lowest degree for R which satisfies these constraints.

Sketch the function R.

Exercise 121

The infected are those that are not susceptible anymore and have not yet recovered. For any t, $S(t) + I(t) + R(t) = 10$

The peak of the epidemic is when the number of infected is highest.

When is the peak of the epidemic? (in weeks)

Radioactive decay

Radioactivity is the property of some substances to decay by transforming into other substances. The intensity of the decay (i.e., the rate of change of the amount of substance with time) is higher when there is more substance: the rate of decay is proportional to the amount of substance.

Exercise 122

Write the equation which expresses the relation given by "the rate of decay is proportional to the amount of substance."

Exercise 123

Write the general equation of decay depending on time.

Exercise 124

The **half-life** is the time it takes for half of the substance to decay. For Carbon-14, the half-life is 5.73 years.

Find the equation which describes the decay of Carbon-14.

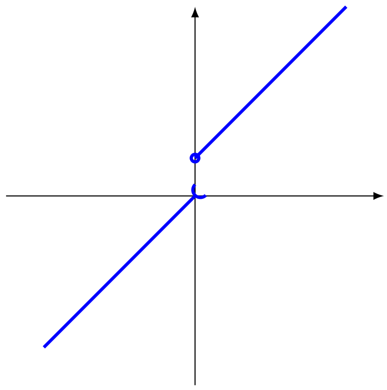
Starting with 100g, how much Carbon-14 is left after one year?

Chapter 14

Limits

Consider the step function $f : x \mapsto \begin{cases} x+1 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$

This function is discontinuous at 0 and if $x \simeq 0_+$ then $f(x) \simeq 1$ and if $x \simeq 0_-$ then $f(x) \simeq 0$.



The point $\langle 0, 0 \rangle$ is not part of the function, yet it is a point of interest for this function since, if one want to draw it, the graph is a straight line open at that point.

> **Definition 30** A **deleted interval** around a is an interval, extending on both sides of a, not containing a.

These and other situations are described by the **limit notation**.

Limit at a point

Informally: the limit of f at a is the value that f should take in order to be continuous at a.

Definition 31 (Limit) Let f be defined **on a deleted interval around** a and x is in that deleted interval.
If there is an observable number L such that

$$x \simeq a \Longrightarrow f(x) \simeq L$$

we say that f **has a limit at** a and write

$$\lim_{x \to a} f(x) = L$$

L is the limit of f at a

Note that by property 4, page 23, if the limit exists, it is unique.
Of course, by this definition, f is continuous at a when

$$\lim_{x \to a} f(x) = f(a)$$

The definition of limit can also be interpreted in the following way:
if f has a limit at a then this limit is the observable neighbour of $f(a + dx)$.

Practice exercise PE29 Answer page 131
Calculate

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3}$$

Using the limit notation, the derivative of f at a can be rewritten in two different ways

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

One sided limits

The function drawn on page 113 is not continuous, but there is also an asymmetry: when $x \simeq 0_+$, then $f(x) \simeq f(0)$, but when $x \simeq 0_-$ we have $f(x) \not\simeq f(0)$. This asymmetry is described by one sided continuity (see definition 10, page 34) and leads to the concept of one sided limit.

Definition 32 The function f **has a limit on the left of** a if there is an observable number L such that

$$x \simeq a_- \implies f(x) \simeq L$$

and write

$$\lim_{x \to a_-} f(x) = L$$

The function f **has a limit on the right of** a if there is an observable number L such that

$$x \simeq a_+ \Rightarrow f(x) \simeq L$$

and write

$$\lim_{x \to a_+} f(x) = L$$

If $\lim\limits_{x \to a_-} f(x) = \lim\limits_{x \to a_+} f(x) = L$ then $\lim\limits_{x \to a} f(x) = L$.
(If the limit on the left and limit on the right are the same, then it is the limit.)

Practice exercise PE30 Answer page 131
Consider the signum function sgn, defined by

$$\text{sgn} : x \mapsto \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Does sgn have a limit at 0?

limit at infinity

The hyperbola, given by $g : x \mapsto \frac{1}{x}$ has a vertical asymptote at $x = 0$ which is not part of the function, yet it is a point of interest for this function.

Also the fact that it has a horizontal asymptote at $y = 0$ leads us to define limits "at infinity".

Asymptotic behaviour can be rewritten using the limit notation and the ∞ sign.
f has a horizontal asymptote at l, on the right if

$$\lim_{x \to \infty} f(x) = l$$

f has a horizontal asymptote at l, on the left if

$$\lim_{x \to -\infty} f(x) = l$$

f has a vertical asymptote at a, on the right if

$$\lim_{x \to a_+} f(x) = \pm\infty$$

f has a vertical asymptote at a, on the left if

$$\lim_{x\to a_-} f(x) = \pm\infty$$

⚠ The fact the we write the the limit **is equal** to infinity does not mean that it reaches infinity – which it does not. For the formal definition of asymptotes, see chapter 3, page 27.

Exercise 125

Calculate the following limits. The answer should be a number, $+\infty$, $-\infty$ or "does not exist"

(1) $\lim\limits_{x\to\infty} \dfrac{6x-4}{2x+5}$

(2) $\lim\limits_{x\to\infty} x^3 - 10x^2 - 6x - 2$

(3) $\lim\limits_{x\to\infty} \dfrac{x^2-x+4}{3x^2+2x-3}$

(4) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+2}}{\sqrt{3x+1}}$

(5) $\lim\limits_{x\to\infty} x - \sqrt{x}$

(6) $\lim\limits_{x\to\infty} \sqrt[3]{x+2}$

(7) $\lim\limits_{x\to 0_-} 1 + \dfrac{1}{x}$

(8) $\lim\limits_{x\to 0} \dfrac{1}{x^2} - \dfrac{1}{x}$

(9) $\lim\limits_{x\to 0} \dfrac{1+2x^{-1}}{7+x^{-1}-5x^{-2}}$

(10) $\lim\limits_{x\to 2} \dfrac{1-x}{2-x}$

(11) $\lim\limits_{x\to 3_+} \dfrac{x+1}{(x-2)(x-3)}$

(12) $\lim\limits_{x\to 3} \dfrac{x+1}{(x-2)(x-3)}$

(13) $\lim\limits_{x\to 1} \dfrac{3x^2+4}{x^2+x-2}$

(14) $\lim\limits_{x\to 2_+} \dfrac{x^2+4}{x^2-4}$

(15) $\lim\limits_{x\to\infty} \sqrt{x^2+1} - x$

(16) $\lim\limits_{x\to -\infty} \sqrt{x^2+1} - x$

(17) $\lim\limits_{x\to\infty} \sqrt{x^2-3x+2} - \sqrt{x^2+1}$

(18) $\lim\limits_{x\to\infty} \sqrt[3]{x+4} - \sqrt[3]{x}$

Integration to infinity

Definition 33 The ∞ symbol in the bounds of an integral indicates a limit.

$$\int_a^\infty f(x)\cdot dx = \lim_{N\to\infty} \int_a^N f(x)\cdot dx$$

This is calculated by taking ultralarge N in $\int_a^N$ and taking the observable part of the result (if it exists and is independent of N).

Exercise 126

Check that the derivative of $x \mapsto \dfrac{x}{x+1}$ is $x \mapsto \dfrac{1}{(x+1)^2}$

Sketch the curve of $f : x \mapsto \dfrac{1}{(x+1)^2}$ for $x > 0$

Calculate the area under f between 0 and 10.

Calculate the area under f between 0 and $+\infty$

Definition 34 If the function to integrate is not defined at the lower bound, then

$$\int_a^b f(x) \cdot dx = \lim_{u \to a_+} \int_u^b f(x) \cdot dx$$

Similarly, if the function to integrate is not defined at the upper bound, then

$$\int_a^b f(x) \cdot dx = \lim_{u \to b_-} \int_a^u f(x) \cdot dx$$

Exercise 127

Evaluate the integrals (after specifying the domain of the function):

(1) $\displaystyle\int_0^1 2x^{-2} \cdot dx$

(2) $\displaystyle\int_{-2}^3 u^{-3} \cdot du$

(3) $\displaystyle\int_{-1}^2 -5(t+1)^{-1/4} \cdot dt$

(4) $\displaystyle\int_0^4 \frac{1}{2\sqrt{x}} \cdot dx$

Summary of this chapter

The limit of a function at a point, gives the value – if it exists – which would make the function continuous at that point.

When the limit of a function exists, both the limit on the left and the limit on the right exist, and they are equal.

The limit at infinity, or a limit at a point being infinite describe asymptotic behaviours.

Integration to infinity or integration to the edge of an open interval are defined as limits.

Chapter 15

Answers to practice exercises

Answer to practice exercise PE1, page 7

Assume there is a fraction equal to $\sqrt{2}$. Then there is a (positive) fraction in simple form such that $\frac{a}{b} = \sqrt{2}$.

Then $\frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$ so a^2 is even.

(We let the reader check that, for natural number n, n^2 is even if and only if n is even.)

Hence a is even. Rewrite $a = 2k$ then $a^2 = 4k^2 = 2b^2$, hence $b^2 = 2k^2$ so b is also even, which contradicts that $\frac{a}{b}$ was in simple form.

Answer to practice exercise PE2, page 11

Think of δ being so small that it can only by seen with a microscope. It could be the size of a bacterium.

(1) δ^2 is even tinier and the important thing here is that it remains microscopically tiny. [1]

If a bacterium can be seen with a microscope, then one still needs a microscope to see two bacteria. So $2 \cdot \delta$ is also tiny.

Since δ has been assumed to be positive and tiny, $-\delta$ is negative, and also tiny.

Tiny numbers are extremely close to zero on both sides.

Note that *tiny* is not the same as small. In mathematics, when numbers are drawn on a line, from left to right, "a is smaller than b" means "a is to the left of b". Here, we use the word "tiny" in an informal way to express closeness to zero.

-10 is smaller than 0.0000000000002 but this last number is tinier than -10

(2) $2 + \delta$ and $2 - \delta$ are "bacterially close" to 2 and it would require the same microscope to see the difference.

They are both extremely close to 2

(3) Just as "smaller" is mathematically defined as being to the left on the numeric line, "larger" is defined as being to the right. To express the idea of being far from zero on either side, we use the word "huge".

[1] If δ is tiny, the its absolute value is less than 1, hence δ^2 is closer to zero still.

Answer to practice exercise PE3, page 11

Instead of microscopes, this time think of telescopes...

(1) The argument is similar to the previous exercise: N^2, $2 \cdot N$ and $-N$ are all huge.

(2) $N + 2$ and $N - 2$ are not far from N but not extremely close (the difference is 2 which is not tiny).

With the δ defined in the previous exercise, $N + \delta$ is extremely close to N.

(3) One divided by a big number is small. So one divided by a huge number is tiny.

(4) $\frac{N}{2}$ is still huge. If you need a telescope to see a planet which is a huge distance away, even half way is still huge.

Answer to practice exercise PE4, page 11

Since $a < b$ we have $a - b < 0$ and $b - a > 0$.

(1) Not tiny divided by tiny is huge. Since $a > 0$ then $\dfrac{a}{a-b} = \dfrac{\text{positive not tiny}}{\text{negative tiny}}$ is a huge negative number.

(2) $\dfrac{a}{b-a}$ is a huge positive number.

(3) A tiny number divided by a not tiny number is tiny. So $\dfrac{b-a}{a}$ is tiny and positive.

(4) $\dfrac{a-b}{a}$ is tiny and negative.

(5) $\dfrac{b-a}{a-b} = -1$

Answer to practice exercise PE5, page 12

The definition states that for each element of the input set there is one and only one corresponding element of the output set.
The condition on the input set and the condition on the output set must both be satisfied. Hence a relation is not a function if one of the conditions fails.
If one (at least one) input does not correspond to an output ore corresponds to more than one output, then it is not a function.

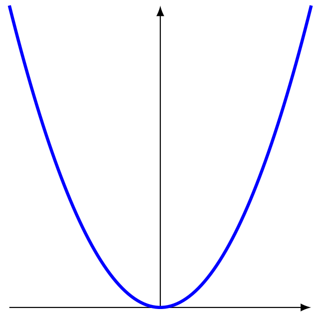
For Example, if the input set is $\{Berlin, Budapest, London, Paris, Vienna\}$, and the output set is $\{Austria, Germany, Hungary, England, United\ States\}$, the relation *is the Capital of* is not a function, since the element $Paris$ of the input set is not associated to any elements of the output set.

> ⚠ Note that the fact that the element $United\ States$ is not associated to any elements of the input set does not prevent the relation from being a function.

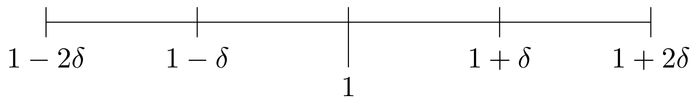
Now, consider the relation *is a multiple of* between the input set $\{8, 25, 27, 49\}$ and the output set $\{2, 3, 5, 7, 9\}$. Again, this relation is not a function, since the element 27 of the input set is associated to more than one element of the output set (namely, 3 and 9).

Answer to practice exercise PE6, page 14

This is the well known parabola:



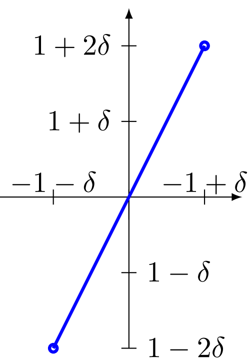
(1) The horizontal line will be as follows:



For the vertical values, we establish a table of values:

x	$f(x) = x^2$
$1-\delta$	$(1-\delta)^2 = 1-2\delta+\delta^2$
1	1
$1+\delta$	$(1+\delta)^2 = 1+2\delta+\delta^2$

Now consider that if δ is $\frac{1}{100}$, then δ^2 is one hundred times smaller, which means that it is almost impossible to distinguish from zero. Things are even more invisible if δ is smaller, hence on the drawing, we cannot distinguish, say, $1+2\delta$ from $1+2\delta+\delta^2$

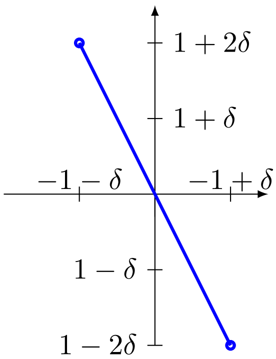


You may also imagine a zoom on a computer screen, the curve seems straighter when you zoom more, until the difference between the actual curve and the straight line is less than a computer screen pixel.

(2) The table of values is as follows:

x	$f(x) = x^2$
$-1-2\delta$	$(-1-2\delta)^2 = 1+4\delta+4\delta^2$
$-1-\delta$	$(-1-\delta)^2 = 1+2\delta+\delta^2$
-1	1
$-1+\delta$	$(-1+\delta)^2 = 1-2\delta+\delta^2$
$-1+2\delta$	$(-1+2\delta)^2 = 1-4\delta+4\delta^2$

and the curve is:

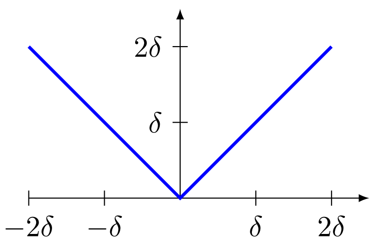


Even though the parabola is a curve, a close up shows a line indistinguishable from a straight line. Of course it is not the same straight line when one zooms on different points.

Answer to practice exercise PE7, page 14

(1)

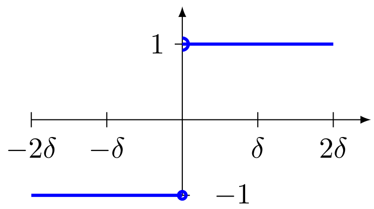
x	$f(x) = \|x\|$
-2δ	2δ
-δ	δ
0	0
δ	δ
2δ	2δ



This time, the "pointed" part remains pointed even after a zoom.

Answer to practice exercise PE8, page 14

x	
-2δ	-1
-δ	-1
0	-1
δ	1
2δ	1



Note that for $x = 0$, the value is -1, not 1 (and not both values). This is symbolised by a full circle where the value is included (closed interval), and a semi-circle where the value is not included (open interval).

This function has a step at zero.

Answer to practice exercise PE9, page 22

(1) Let $x = N$ be ultralarge, and $y = N + \frac{1}{N}$ so $x \simeq y$ but $x^2 = N^2 \not\simeq N^2 + 2 + \frac{1}{N^2} = y^2$

(2) Let h be ultrasmall, then let $x = h$ and $y = h^2$. Then $x \simeq 0$ and $y \simeq 0$ hence $x \simeq y$. Then $\frac{1}{h}$ and $\frac{1}{h^2}$ are both ultralarge and $\frac{1}{h^2} - \frac{1}{h} = \frac{1}{h}(\frac{1}{h} - 1)$. By ultracomputation (page 21), this is ultralarge, hence $\frac{1}{x} \not\simeq \frac{1}{y}$

Answer to practice exercise PE10, page 22

(1) As $\frac{1}{\varepsilon}$ is ultralarge $1 + \frac{1}{\varepsilon}$ is ultralarge.

(2) We have $\frac{\sqrt{\delta}}{\delta} = \frac{1}{\sqrt{\delta}}$ which is ultralarge.

(If $\delta < c$ for any observable c, then $\sqrt{\delta} < \sqrt{c}$ and $\sqrt{\delta} \simeq 0$ hence $\frac{1}{\sqrt{\delta}}$ is ultralarge.)

(3) Maybe surprisingly, this is ultrasmall. To see this we multiply and divide by the conjugate:

$$\begin{aligned}
\sqrt{H+1} - \sqrt{H-1} &= \frac{(\sqrt{H+1} - \sqrt{H-1})(\sqrt{H+1} + \sqrt{H-1})}{\sqrt{H+1} + \sqrt{H-1}} \\
&= \frac{(H+1) - (H-1)}{\sqrt{H+1} + \sqrt{H-1}} \\
&= \frac{2}{\sqrt{H+1} + \sqrt{H-1}}
\end{aligned}$$

H is assumed positive, its square root is also a positive ultralarge. The sum of 2 positive ultralarge numbers is ultralarge hence the quotient is ultrasmall.

(4) $\dfrac{H+K}{HK} = \dfrac{1}{K} + \dfrac{1}{H}$ is ultrasmall.

(5) $\dfrac{5+\varepsilon}{7+\delta} - \dfrac{5}{7} = \dfrac{35+7\varepsilon-35-5\delta}{49+7\delta} = \dfrac{\overbrace{7\varepsilon-5\delta}^{\simeq 0}}{\underbrace{49+7\delta}_{\simeq 49}}$ is ultrasmall or zero.

(6) $\dfrac{\overbrace{\sqrt{1+\varepsilon}-2}^{\simeq -1}}{\underbrace{\sqrt{1+\delta}}_{\simeq 1}} \simeq -1$, hence not ultralarge and not ultrasmall.

Answer to practice exercise PE11, page 22

(1) Take $\varepsilon = \delta$ then $\dfrac{\varepsilon}{\delta} = 1$

(2) Take $\delta = \varepsilon^2$, then $\dfrac{\varepsilon}{\delta} = \dfrac{1}{\varepsilon}$ is ultralarge.

(3) Take $\varepsilon = \delta^2$, then $\dfrac{\varepsilon}{\delta} = \delta$ is ultrasmall.

Answer to practice exercise PE12, page 32

Vertical asymptote of the form $x = c$, horizontal asymptote of the form $y = b$, oblique asymptote of the form $y = ax + b$.

(1) $y = x$

(2) $y = 1$, $x = 0$, $x = 4/3$

(3) $\begin{cases} y = x & \text{if } x > 0 \\ y = -x & \text{if } x < 0 \end{cases}$

(4) $y = \sqrt{1/3}$, $x = \sqrt[4]{1/3}$

(5) $\begin{cases} y = 0 & \text{if } x < 0 \\ y = 1 & \text{if } x > 0 \end{cases}$

Answer to practice exercise PE13, page 43

(1) 10

(2) -70

(3) 10

(4) 20

Answer to practice exercise PE14, page 43

Assuming f is defined around x, we write

$$f(x+dx) = (x+dx)^2 + 3(x+dx) = x^2 + 2x \cdot dx + dx^2 + 3x + 3dx$$

then

$$f(x+dx) - f(x) = 2x \cdot dx + dx^2 + 3dx$$

divide by dx

$$\frac{f(x+dx) - f(x)}{dx} = 2x + dx + 3 \simeq 2x + 3$$

Answer to practice exercise PE15, page 43

(1)
$$\frac{(x+dx)^2 - x^2}{dx} = \frac{x^2 + 2x \cdot dx + dx^2 - x^2}{dx} = 2x + dx \simeq 2x$$

(2)
$$\frac{(x+dx)^3 - x^3}{dx} = \frac{x^3 + 3x^2dx + 3x \cdot dx^2 + dx^3 - x^3}{dx} = 3x^2 + 3x \cdot dx + dx^2$$

the quantity dx^2 is ultrasmall. But for $3x \cdot dx$ we need to recall property 2 (1), which states that an observable multiplied by an ultrasmall is is equal to an ultrasmall. x is observable since we are differentiating at that point (it is a parameter of the formula) and dx is ultrasmall.

Hence we can conclude
$$3x^2 + 3x \cdot dx + dx^2 \simeq 3x^2$$

Answer to practice exercise PE16, page 58

(1) $f'(x) = 20x^3 + 3x^2 - 4x$

(2) $g'(x) = 10\sqrt{3}x$

(3) $h'(x) = -\dfrac{x^4 + 4x^3 - 3x^2 + 10x + 10}{(x^3 - 5)^2}$

(4) $j'(x) = 20x^3 - \dfrac{6x - 2}{(3x^2 - 2x + \pi)^2}$

(5) $k'(x) = 0$

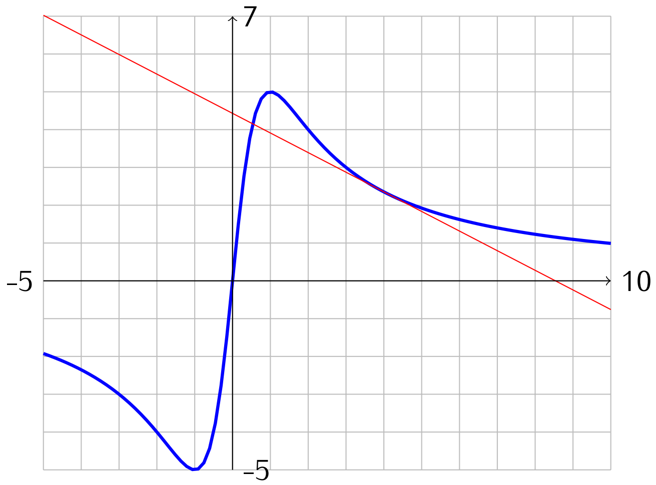
(6) $l'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} - \dfrac{3}{x^4} - \dfrac{4}{x^5}$

(7) $m'(x) = \dfrac{(x^2 + x + 1)(3x^2 + 2x) - (x^3 + x^2)(2x + 1)}{(x^2 + x + 1)^2} = \dfrac{x(x^3 + 2x^2 + 4x + 2)}{(x^2 + x + 1)^2}$

Answer to practice exercise PE17, page 59

Answer to practice exercise PE18, page 59

Tangent line is $y = -\dfrac{4}{5}x + \dfrac{27}{5}$



Answer to practice exercise PE19, page 59

(1) $3x^2 + 2x + 2$

(2) $-3x^2 + 4x - 2$

(3) $x^2 - 5x + 6$

(4) $(x-2)^2$

(5) $\dfrac{x(x+4)}{(x+2)^2}$

(6) $\dfrac{x^2 + 2x - 8}{(x+1)^2}$

(7) $\dfrac{4x^2+4x-3}{(2x+1)^2}$

(8) $-\dfrac{x^2+6x+5}{(x+3)^2}$

(9) $\begin{cases} 1 & \text{if } x>2 \\ -1 & \text{if } x<2 \\ \text{not differentiable} & \text{if } x=2 \end{cases}$

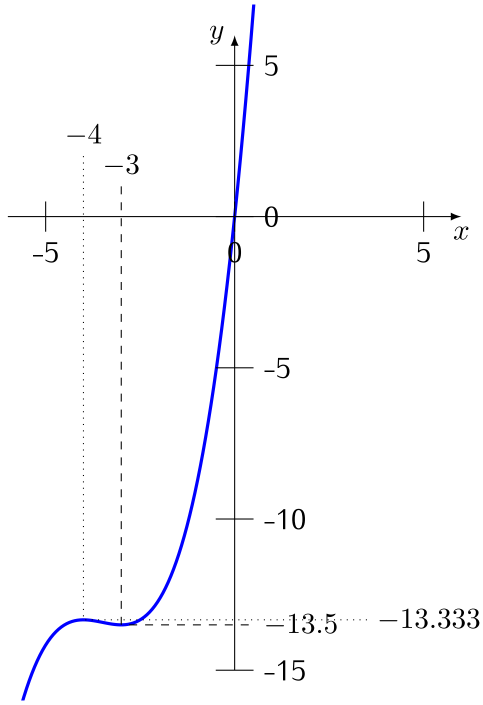
(10) $\begin{cases} \dfrac{x(x+4)}{(x+2)^2} & \text{if } x\geq 0 \\ \dfrac{-x(x-4)}{(x-2)^2} & \text{if } x\leq 0 \end{cases}$

(11) $\dfrac{x^2+2x+2}{(x+1)^2}$

(12) $\begin{cases} 3x^2-12x+11 & \text{if } x\in]1,2[\cup]3,\infty[\\ -3x^2+12x-11 & \text{if } x\in]-\infty,1[\cup]2,3[\\ \text{not differentiable} & \text{if } x\in\{1,2,3\} \end{cases}$

Answer to practice exercise PE20, page 60

$f(x)=x\left(\dfrac{1}{3}x^2+\dfrac{7}{2}x+12\right)$
$\mathcal{S}=\{0\}$
$f'(x)=x^2+7x+12=(x+3)(x+4)$
$\mathcal{S}'=\{-3,-4\}$

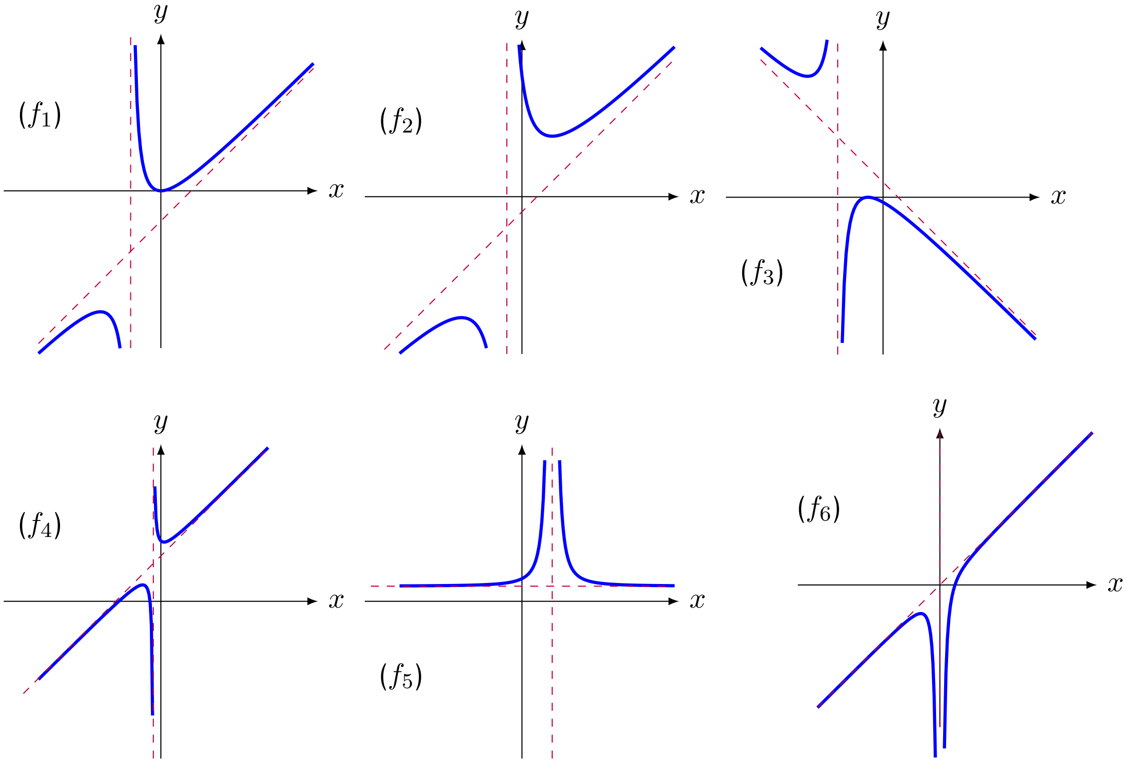


Answer to practice exercise PE21, page 60

(1) $t_a : x\mapsto -12x-3$

(2) $t_b : x\mapsto \dfrac{5}{7}x-\dfrac{2}{7}$

Answer to practice exercise PE22, page 64



Answer to practice exercise PE23, page 80

(1) 1 Use $x = 1 - u^2$

(2) undefined – for $u > 1$ we have the square root of a negative number.

(3) $\frac{8(\sqrt{2}+1)}{15}$ Use $u = 1 + \sqrt{x}$

(4) $\frac{50}{309}$ Use $u = t^2 + 3$

(5) $\frac{195}{8}$ Use $u = x^2 + 2$

(6) $\frac{2}{45}$ Use $u = 4 - x^3$

(7) $-\sqrt{6} + 2\sqrt{2}$ Use $u = 1 + \frac{1}{t}$

Answer to practice exercise PE24, page 80

(Integration constant to be added)

- $F_a : x \mapsto x^5 - x^2 + 4x$
- $F_b : x \mapsto \dfrac{1}{4}x^4 - \dfrac{5}{3}x^3 + \dfrac{3}{2}x^2 - 2x$
- $F_c : x \mapsto x^2 - x$
- $F_d : x \mapsto \dfrac{1}{4}x^5 - \dfrac{1}{4}x^3 + \dfrac{5}{4}x^2 + \dfrac{3}{2}x$
- $F_e : x \mapsto x^2 + x + \dfrac{1}{x}$
- $F_f : x \mapsto 3x - \dfrac{2}{x} + \dfrac{5}{2x^2}$
- $F_g : x \mapsto \dfrac{x^4}{4} - \dfrac{1}{x}$
- $F_h : x \mapsto \dfrac{3}{4}\sqrt[3]{x^4} + \dfrac{3}{2}\sqrt[3]{x^2}$
- $F_i : x \mapsto 2\sqrt{x} + \dfrac{2}{3}\sqrt{x^3}$
- $F_j : x \mapsto \dfrac{1}{3}(x+1)^3$
- $F_k : x \mapsto (3x-2)^5$

- $F_l : x \mapsto \frac{1}{8}(2x+1)^4$
- $F_m : x \mapsto -\frac{1}{12}(3-x)^{12}$
- $F_n : x \mapsto -\frac{1}{20}(3-4x)^5$
- $F_o : x \mapsto \frac{2}{9}\sqrt{(3x-2)^3}$
- $F_p : x \mapsto 2\sqrt{x-1}$
- $F_q : x \mapsto -\frac{1}{3}(3-x^2)^6$
- $F_r : x \mapsto \frac{1}{5}(x^2-3x+1)^5$
- $F_s : x \mapsto \frac{1}{3}(x^3-2x^2+x-3)^3$
- $F_t : x \mapsto \frac{2}{3}(4x^2-5x)^3$
- $F_u : x \mapsto \frac{1}{8}(3x^2-2x+5)^4$
- $F_v : x \mapsto -\frac{1}{x^2+1}$
- $F_w : x \mapsto -\frac{1}{x^2+x+3}$
- $F_x : x \mapsto \frac{1}{3}\sqrt{(x^2+1)^3}$
- $F_y : x \mapsto 2\sqrt{9+x^3}$
- $F_z : x \mapsto \frac{2}{3}(x^3+x+2)\sqrt{x^3+x+2}$

Answer to practice exercise PE25, page 95

If $u = \frac{t}{a}$ (t and u being variables, a is a constant) we have $du = dt \cdot a$.
If $t = a$ we have $u = \frac{a}{a} = 1$ and if $u = a \cdot b$ we have $u = \frac{t}{a} = \frac{a \cdot b}{a} = b$
Replacing each term we get

$$\int_a^{a \cdot b} \frac{1}{t} = \int_1^b \frac{1}{a \cdot u} a \cdot du = \int_1^b \frac{1}{u} du$$

Answer to practice exercise PE26, page 103

(1) $1/6$

(2) $1/16$

(3) 0

(4) $-1/4$

(5) $1/2$

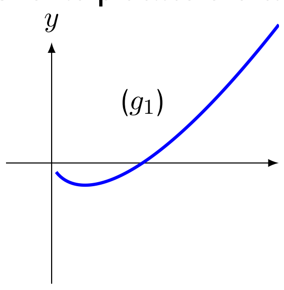
(6) -1

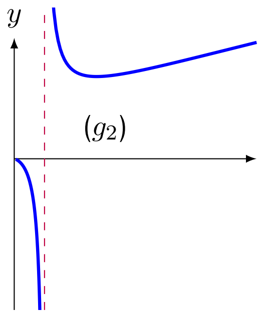
(7) 5

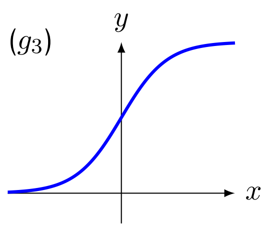
(8) ultralarge

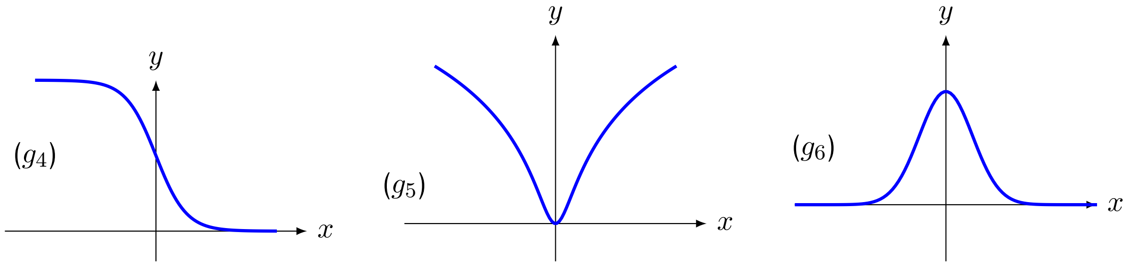
(9) 2

Answer to practice exercise PE27, page 106









Answer to practice exercise PE28, page 108

(1) We check that $p(t) = C \cdot e^{kt}$ is a solution to

$$p'(t) = k \cdot p(t)$$

(for constants C and k to be found).

$$p'(t) = (C \cdot e^{kt})' = C \cdot (e^{kt})' = C \cdot k \cdot e^{kt}$$

by the chain rule applied to e^{kt}

Note that

$$C \cdot k \cdot e^{kt} = k \cdot p(t)$$

so this function is indeed a solution for the population growth problem.

If $t = 0$, the population is 7.8 billion or $7.8 \cdot 10^9$

This is called an initial condition.

$p(0) = C \cdot e^{k \cdot 0} = C \cdot e^0 = C$ hence $C = 7.8 \cdot 10^9$

If $t = 1$ (one year later) the population has increased by 1.05% hence is then $7.8819 \cdot 10^9$

$$p(1) = C \cdot e^k = C + C \cdot 1.05\% = C \cdot (1 + 1.05\%) = C \cdot 1.0105$$

Note that it is possible to divide both sides by C: the only thing that influences k is the annual rate of change: 1.05%

We therefore have $e^k = 1.0105$

To find k we use the inverse function of the exponential and $k = \ln(1.0105)$

Hence the final formula is

$$p(t) = 7.8 \cdot 10^9 \cdot e^{\ln(1.0105)t}$$

which simplifies to

$$p(t) = 7.8 \cdot 10^9 \cdot 1.0105^t$$

(2) The doubling time requires to solve

$$p(t) = C \cdot 1.0105^t = 2 \cdot C$$

Again, dividing both sides by C, we need to solve

$$1.0105^t = 2$$

Using logarithms again to find the exponent:

$$\ln(1.0105^t) = \ln(2)$$

hence

$$t \cdot \ln(1.0105) = \ln(2)$$

which leads to $t = \frac{\ln(2)}{\ln(1.0105)} \approx 66.36$

If the rate of change remains constant, the world population will double approximately every 66 years

Answer to practice exercise PE29, page 114

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3}$$

$\lim_x \to 3$ means that we take $x \simeq 3$ and write $x = 3 + dx$

$$\frac{2 \cdot (3 + dx)^2 - 7 \cdot (3 + dx) + 3}{3 + dx - 3} = \frac{5dx + 2dx^2}{dx} = 5 + 2dx \simeq 5$$

hence

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3} = 5$$

(Notice that the limit **is equal** to 5 and not ultraclose. The limit is the value that the expression is ultraclose to.)

The expression

$$\lim_{x \to 3} \frac{2x^2 - 7x + 3}{x - 3}$$

is equivalent to

$$\lim_{h \to 0} \frac{2 \cdot (3 + h)^2 - 7 \cdot (3 + h) + 3}{h}$$

Answer to practice exercise PE30, page 115

$$\text{sgn} : x \mapsto \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

$\lim\limits_{x \to 0_-} \text{sgn}(x) = -1 \neq \text{sgn}(0)$ and $\lim\limits_{x \to 0_+} \text{sgn}(x) = 1 \neq \text{sgn}(0)$

so not only are the limits not equal to the value of the function, but they are different on the left and on the right. Hence sgn is not continuous at 0

Bibliography

[1] K. Hrbacek, O. Lessmann, and R. O'Donovan. Analysis with ultrasmall numbers. American Mathematical Monthly, 117(9), November 2010.

[2] K. Hrbacek, O. Lessmann, and R. O'Donovan. Analysis With Ultrasmall Numbers. CRC Press, 2015.

[3] H. J. Keisler. Foundations of Infinitesimal Calculus. University of Wisconsin, 2007.

[4] H. J. Keisler. Elementary Calculus, An Infinitesimal Approach. University of Wisconsin, 2013.

[5] R. O'Donovan. Calculus in high school with infinitesimals, the pedagogical advantages. Oral Presentation, AMS-UMI meeting, Pisa, June 2002.

[6] R. O'Donovan. What is the value of $f'(2+\delta)$? Oral Presentation, NSM meeting, Aveiro, July 2004.

[7] R. O'Donovan. An additive group containing infinities. In Seminarberichte aus dem Fachbereich Mathematik, volume 76, pages 159–164. FernUniversität in Hagen, 2005.

[8] R. O'Donovan. Pre-university analysis. In V. Neves and I. Van den Berg, editors, The Strength of Nonstandard Analysis. Springer, 2007.

[9] R. O'Donovan. Teaching analysis with ultrasmall numbers. Mathematcis Teaching-Research Journal, 3(3):1–22, August 2009.
http://www.hostos.cuny.edu/departments/math/mtrj/archives
/volume3/issue3/Donovan.pdf.

[10] R. O'Donovan. Teaching analysis with ultrasmall numbers. In Analisi non standard per le scuole superiori, pages 29–44, 2016.

[11] R. O'Donovan. Nonstandard analysis in the classroom: ultrasmall numbers vs. limits: a comparison. In Analisi non standard per le scuole superiori, pages 35–44, 2017.

[12] R. O'Donovan. Delta functions and differentiation of discontinuous functions. In Analisi non standard per le scuole superiori, pages 55–69, 2018.

[13] R. O'Donovan and J. Kimber. Nonstandard analysis at pre-university level. In D. R. N.J. Cutland, M. Di Nasso, editor, Nonstandard Methods and Applications in Mathematics, Lecture Notes in Logic 25. Association for Symbolic Logic, 2006.

[14] Y. Péraire. Théorie relative des ensembles internes. Osaka Journal of Mathematics, 29:267–297, 1992.

[15] A. Robert. Analyse non standard. Presses polytechniques romandes, 1985.

[16] K. D. Stroyan. Calculus, The Language of Change. Academic Press, 1993.

[17] K. D. Stroyan. Mathematical Background: Foundations of Infinitesimal Calculus. Academic Press, 1997.

List of Theorems

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