Construction of Number Systems Including Ultralarge and Ultrasmall Numbers

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December 11, 2022

This paper examines how one goes from extra axioms about sets to extra properties of real numbers. The classical construction of real numbers is assumed and the proof that all sets of real numbers bounded above have a least upper bound being a proof of classical mathematics will not be given here but can be found in [4] and [5]. (1)

For a fuller discussion of these axioms, see [3], and for the fact that they do not introduce contradictions, see [2] (downloadable here). This last paper deals with model theory.

These axioms about set theory were specifically designed to produce results such as the existence of ultralarge and ultrasmall numbers, the existence of the observable neighbour and closure.

1 Principles of Analysis with Ultrasmall Numbers

The axioms of set theory are denoted by ZFC (Zermelo and Fraenkel with axiom of choice) as given in [1]. Three axioms are added: Idealisation, Transfer and Standardisation.

Idealisation is used to show the existence of ultrasmall and ultralarge numbers.

Transfer yields the closure principle.

Standardisation produces the observable neighbour principle.

Definition 1. *The* ***context***(2) *of a property, function or set, is the list of parameters used in its definition.*

The word "observable" always refers to a context.

The context can be a single parameter or even empty.

(1) The fact that these extra axioms do not introduce contradictions in set theory is treated in [2] and belongs to model theory.

(2) In pedagogical approaches, the concepts of context and observability are sometimes merged – for the sake of introducing one word less.

Observability Principle

- Every set is observable relative to its own context.
- $\emptyset$ is observable relative to every context.
- Two sets a and b will always have a common context. If a is not observable in the context of b, then b is observable in the context of a.

If a set is observable relative to every context, we say that it is always observable.

In the following, **C** refers to the context and $\forall^{\mathbf{C}} A$ means: for any observable A.

Idealisation:
For P being a property not referring to observability:

$$(\forall \overline{x})(\forall^{\mathbf{C}} A)[(\forall^{\mathbf{C}} a \in P^{\mathbf{fin}} A)(\exists y)(\forall x \in a)\, \mathcal{P}(x, y, A, \overline{x}) \leftrightarrow (\exists y)(\forall^{\mathbf{C}} x \in A)\mathcal{P}(x, y, A, \overline{x})]$$

This reads:

IF: for any list of variables, x_i, for any observable A, for any observable finite subset a of A, there is a y such that for any x in a, the property P (referring to the list of x_i, x, y and a) holds
THEN there is a y such that the property holds for all observable x in A

Rewritten with less variables:

Idealisation

Given a set A and a property P not referring to observability.
If for every observable finite subset a of A there is a y such that the property $P(x, y, A)$ holds for every $x \in a$, then there is a y such that $P(x, y, A)$ holds for every observable $x \in A$.

Theorem 1 (Existence of non observable numbers). *There exist natural numbers which are not observable.*

Proof. The set A used in the idealisation principle is $\mathbb{N}$.

The property not referring to observability is "$y \in \mathbb{N}$ and $y > x$".

For every finite observable subset $B \subset \mathbb{N}$, there is a y such that for every $x \in B$, the property "$y \in \mathbb{N}$ and $y > x$" holds (take $y = \max\{B\} + 1$)

Hence, by idealisation, there is a y such that "$y \in \mathbb{N}$ and $y > x$" holds for all observable $x \in \mathbb{N}$. Obviously, this y cannot be observable. □

If we restrict B to sets defined without reference to observability, we get that there are natural numbers which are not always observable. If we then introduce such a number in the definition of B (such as all finite sets containing k), we get numbers which are less observable still.

Transfer:
For $\mathcal{P}(x_1, \ldots, x_k)$ is any statement not referring to observability.

$$(\forall \mathbf{C})(\forall^{\mathbf{C}} x_1) \ldots (\forall^{\mathbf{C}} x_k)\ (\mathcal{P}^{\mathbf{C}}(x_1, \ldots, x_k) \leftrightarrow \mathcal{P}(x_1, \ldots, x_k))$$

This reads:

For any context, if a statement holds when all variables are observable, then the statement holds for all values of the variables.

Theorem 2. ***Closure***

If there is an x such that $P(x)$, then there is an observable x such that $P(x)$.

Proof. Assume that $\mathcal{P}(x)$ is a statement not referring to observability and $(\exists x)\ \mathcal{P}(x)$ holds. By Transfer then also $(\exists^{\mathbf{C}} x)\ \mathcal{P}^{\mathbf{C}}(x)$ holds. Fix x such that x is observable and $\mathcal{P}^{\mathbf{C}}(x)$ holds. By Transfer once more, the statement $\mathcal{P}(x)$ holds as well, and therefore we conclude that $(\exists^{\mathbf{C}} x)\ \mathcal{P}(x)$. □

Closure – contrapositive form

If $P(x)$ holds for all observable x, then it holds for all x.

If it did not hold for some x, then it would not hold for some observable x and we assume the contrary.

Consequence: it two observable sets contain the same observable elements, then they are the same sets.

Standardization:

$$(\forall \mathbf{C})(\forall \overline{x})(\forall x)(\exists^{\mathbf{C}} y)(\forall^{\mathbf{C}} z)(z \in y \leftrightarrow z \in x\ \wedge\ \mathcal{P}(z, x, \overline{x}; \mathbf{C}))$$

where $\mathcal{P}(z, x, \overline{x}; \mathbf{C})$ is any statement referring to the context or not referring to observability.

This reads:

For any context, for any list of variables x_i, for any x, there is an observable set y such that for any z, this z belongs to y iff it belongs to x and satisfies a given property $\mathcal{P}$.

In reduced form:

Standardisation Consider a context and a set B which is not observable.
Then there is an observable set A such that all observable elements of B are exactly the observable members of A.

Note that if B contains no observable number, then $A = \emptyset$

Theorem 3. *The standardisation of a set is unique.*

Proof. Assume there are two standardisations A and B of a set C. A and B have same observability and contain the same observable elements, hence by closure all their elements are the same. □

Definition 2 (Ultralarge numbers)**.** *Relative to a context; if a number is greater in absolute value than any observable number, then it is ultralarge.*

Definition 3 (Ultrasmall numbers)**.** *Relative to a context; if a number is smaller in absolute value than any non zero observable number, then it is ultrasmall.*

Theorem 4 (**Observable neighbour**). *If x is not ultralarge, then there is a unique observable real number a and $h \simeq 0$ such $x = a + h$.*

Note that $h \simeq 0$ stands for h is ultrasmall or zero (ultraclose to zero).

Proof. Fix a context and a real number x not ultralarge (not necessarily in the context). Wlog assume $x > 0$. Consider the set $B = \{u \in \mathbb{R} \mid u \leq x\}$. (the context of this set is given by k). This set has a unique standardisation A (not empty since it contains 0) and is bounded above by an observable number since x is not ultralarge. Therefore A has a least upper bound a – which is observable (by closure).

We now show that $x \simeq a$, so a is the observable neighbour of x.

If not, then $|a - x| > s > 0$ for some observable s. This means that either $x > a+s$ or $x < a-s$. In the first case, $a+s \in A$, contradicting $a = \sup A$. In the second case, $a-s$ is an upper bound on A, again contradicting $a = \sup A$.

We now show that $x \simeq a$. A contains all observable elements of B

If not, then for some positive observable s either $x > a + s$ or $x < a - s$. In the first case, since $a + s$ is observable and in B, we have $a + s \in A$, contradicting $a = \sup A$.

In the second case, $a - s$ is an upper bound on A, again contradicting $a = \sup A$. □

Theorem 5. *Let n be an integer; if n is not observable, then n is ultralarge.*

Proof. Assume that n is not ultralarge. By the Observable Neighbour Principle, there is an observable r such that $n \simeq r$. But n is the unique integer in the interval $[r - 0.5,\ r + 0.5)$, hence n is observable by Closure, contradicting our assumption. □

This can be rephrased:

Theorem 6. *If $k, n \in \mathbb{N}$, $k \leq n$, and n is observable, then k is observable.*

Theorem 7. *If A is an observable finite set, then each element of A is observable.*

Proof. To say that A is finite means that there is a sequence $\langle a_1, \ldots, a_n \rangle$, $n \in \mathbb{N}$, such that $A = \{a_1, \ldots, a_n\}$. This is a statement with parameter A. By Closure, there is an observable sequence with this property. The number n is uniquely determined by the sequence (it is the largest element of its domain); hence it is also observable. By theorem 6, every $i \leq n$ is observable. Therefore, for any i, a_i, the unique value of the sequence at i, is observable. □

Theorem 8. *If an observable set contains non observable elements, then it is an infinite set.*

This is simply the contrapositive of theorem 7 and the example we have already seen is that $\mathbb{N}$ is always observable and yet contains nonobservable items and is, of course, infinite.

Lemma 1. *Let $a \in \mathbb{Q}$.*
$\frac{1}{a}$ *is as observable as a*

Proof. The context is given by a. Since there is a number equal to $\frac{1}{a}$, by closure, there is an observable such number. And by uniqueness of the result, $\frac{1}{a}$ is observable. $\square$

Theorem 9. *Relative to some context; if $M \in \mathbb{N}$ is ultralarge, then $\frac{1}{M}$ is an ultrasmall rational number.*

Proof. Assume, for a contradiction, that $\frac{1}{M}$ is not ultrasmall i.e., there is a rational observable non zero a such that $0 < a < \frac{1}{M}$. But then $\frac{1}{a} > M$ so there would be an observable number greater than an ultralarge natural number: a contradiction. $\square$

The existence of ultrasmall and ultralarge real numbers is an immediate consequence of their existence in $\mathbb{Q}$.

Internal or standard view?

A question sometimes asked about these ultralarge and ultrasmall numbers is whether they are the *same* numbers as the ones used by those who do not use the concept of observability. We first stress that this question is not mathematical but philosophical.

The internal view is that they are the same.

The standard view is that idealisation *produces* new objects.

The question is about a philosophical interpretation of "then there exists". In the proof that in $\mathbb{N}$ there exists ultralarge numbers, it can be interpreted as "there already exists" (internal view) or "now there also exist" (standard view).

Whichever view is adopted, the mathematics are the same.

For teaching analysis at introductory level, there is no conflicting point of view: it is all new, hence the internal view can be favoured. For the trained mathematician, the standard view is sometimes more comfortable.

References

[1] Karel Hrbacek and Thomas Jech. *Introduction to Set Theory. Third Edition, Revised and Expanded.* Marcel Dekker, Inc., 1999.

[2] Karel Hrbacek *Consistency of RBST (2014)* downloadable on this site

[3] Karel Hrbacek, Olivier Lessmann and Richard O'Donovan *Analysis with ultrasmall numbers* Chapman and Hall, CRC Press, 2014.

[4] Michael Spivak *Calculus. Third edition.* Publish or Perish Inc. 1994.

[5] Howard Levi. *Elements of Algebra, fourth edition.* Chelsea Publishing, 1961.