Two simple examples of external functions

December 16, 2022

Whether working at a research level or at the introductory level, the following definitions and principles are given, yet the wording may differ.

The context of a function, set or property is the list of parameters used in their definition.

The context does not cover dummy variables. When writing $f(x)$ we write a statement about f (its defining parameters and x)

$f'(x)$ is defined by using $dx \simeq 0$ where the context is given by f and x but the statement is not about dx which is just a convenient temporary variable. (Similarly, in the classical definition, the derivative is not about ε or δ).

Definition Principle Sets, functions and properties may be defined either by not referring to observability or be the use of the contextual symbol $\simeq$.

Sets, functions and statements that satisfy the Definition Principle are called **Internal**.

Internal functions are the ones that behave in the same way than the functions of classical analysis.

In high school handouts it is customary to restrict reference to observability by introducing the $\simeq$ symbol, which by its definition refers only to the context. Since no other observability related symbolism is given, non contextual references are avoided by what amounts to a syntactic rule.

External objects are those that are defined without following the contextual restriction and can behave in very disturbing ways! The way to not respect the contextual restriction is to have a reference to a fixed observability

(for convenience, we will use the terminology that if x is always observable, we say the x is standard.)

Let

$n_p(x)$ be the observable neighbour of x in the context of p

Then $n_0(x)$ is the standard neighbour of x.

The external set $E = \{x \in \mathbb{R} \mid n_0(x) = x\}$ is the collection of all standard elements of $\mathbb{R}$. It is bounded above (by any ultralarge number) but has no least upper bound: there is no last non ultralarge number, or no first ultralarge number.

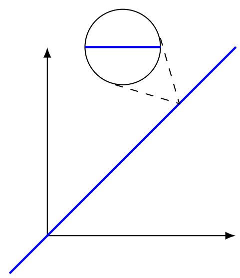
The external set $F = \{x \in \mathbb{R} \mid n_0(x) = 0\}$ is the collection of all numbers ultraclose to zero. And is bounded above by 1 but also has no l.u.b.

Let

$$f : x \mapsto n_0(x)$$

$f(x)$ is about x but the reference to observability does not refer to x. The reference to the standard level, regardless of the observability of x, makes it an external function.

At the standard level, it looks like $x \mapsto x$ but zooming in shows a horizontal line. And because there is no right or left boundary to these horizontal lines, there is no point where a discontinuity can be defined.

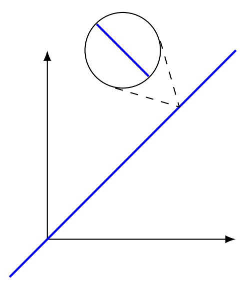


There is no x such that $f(x)$ is not standard so the intermediate value theorem is not valid!

Let

$$g : x \mapsto 2 \cdot n_0(x) - x$$

This one, which we call the updown function, looks like



Its slope is everywhere equal to -1. The function is nowhere discontinuous, satisfies the intermediate value theorem, but not the mean value theorem. It is locally decreasing and globally increasing.

These external objects can be fun but we strongly recommend that they should not be shown in the classroom! Rather, these examples are here to show the importance of restricting reference to observability to the use of the contextual notational symbol $\simeq$