Equivalence of definitions of limits

One can very well consider that it is possible to go back to properties defined without using observability by "forgetting" these extra properties. Then all that is needed is a proof that the limit defined in the classical ε-δ way and the one defined using ultracloseness are equivalent.

Definition 1. *The function f has a limit at a if there is an L, such that*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \quad |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Definition 2. *The function f has a limit at a if there is an observable L, such that*

$$\forall x \quad x \simeq a \Rightarrow f(x) \simeq L$$

Note that in the second definition, if there is an L such that $\forall x \quad x \simeq a \Rightarrow f(x) \simeq L$, then by closure there is an observable such L.

Theorem 1. *Definition (1) $\iff$ Definition (2)*

(2) $\Rightarrow$ (1)
Assume $x \simeq a \Rightarrow f(x) \simeq L$
Observability is given by f, a and L
Choose ε and extend observability to f, a, L and ε
Let $\delta \simeq 0$. If $|x - a| < \delta$ then $x \simeq a$.
Under the initial assumption, this implies that $f(x) \simeq L$ or $|f(x) - L| \simeq 0$
But ε is observable, hence $|f(x) - L| < \varepsilon$.
Since this is true for any ε, we have

$$(x \simeq a \Rightarrow f(x) \simeq L) \Longrightarrow (\forall \varepsilon > 0, \exists \delta > 0 \quad |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

(1) $\Rightarrow$ (2)
Assume $\forall \varepsilon > 0, \exists \delta > 0 \quad |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
Observability is given by f, a and L. Choose ε and extend observability to f, a, L and ε

If there exists a δ having a given property, by closure, there is an observable δ having the same property.

Hence, for all $x \simeq a$ we have $|x - a| < \delta$, which implies by initial assumption $|f(x) - L| < \varepsilon$ i.e.,

For all observable ε, $x \simeq a \Rightarrow |f(x) - L| < \varepsilon$, hence $f(x) \simeq L$, therefore

$$(\forall \varepsilon > 0, \exists \delta > 0 \quad |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon) \Longrightarrow (x \simeq a \Rightarrow f(x) \simeq L)$$