External sets.

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1 External Sets

The textbook "*Analysis with ultrasmall numbers*" (AUN) scrupulously maintains the viewpoint that only internal statements can be used to define sets and functions. It gives several examples of statements that are not internal (henceforth, we shall call them **external**) and for which there is no set of objects with the property specified by the statement. We hope that AUN demonstrates that a lot of interesting mathematics using infinitesimal methods can be carried out in this "internal framework." Nevertheless, many external statements, such as "n is a standard natural number," are used throughout the book, and are considered to be definitely true or false for each particular value of their parameters. It is very natural to talk about the "set"

$$\mathbf{m}_p(0) = \{x : x \simeq 0 \text{ relative to } p\},$$

the "function"

$$\mathbf{n}_p : x \mapsto \mathbf{n}_p(x)$$

that assigns to each real number x not ultralarge relative to p its observable neighbor relative to p, and other similar concepts. It turns out that this causes no difficulties, as long as we maintain strict distinction between the "ordinary" numbers, sets, and functions we were concerned with in AUN, and the new, "exotic" sets and functions defined by external statements and called **external sets and functions** henceforth. In this section we accomplish it by using boldface letters for external sets and functions.

The modern theory of infinitely small and infinitely large quantities was developed by Abraham Robinson in the 1960's. It became known as "Nonstandard Analysis," from the title of Robinson's book. One of the characteristic features of Robinson's framework for Nonstandard Analysis is the use of external objects from the very beginning of the development of the subject. The goal of these notes is to provide a bridge from the "internal framework" used in AUN to the writings of nonstandard analysts.

In the rest of this section we define some fundamental concepts from the literature of Nonstandard Analysis, and reformulate a few of our definitions and theorems in this terminology. The proofs are simple exercises. The next section examines the "external framework" of Nonstandard Analysis in a more systematic manner.

Definition 1
Relative to a context p:

(1) *The external set* $\mathbf{m}_p(a) = \{x \in \mathbb{R} : x \simeq a$ *relative to* $p\}$ *is called the* ***monad*** *of* a *relative to* p.

(2) *The external set* $\mathbf{g}_p(a) = \{x \in \mathbb{R} : x \sim a$ *relative to* $p\}$ *is called the* ***galaxy*** *of* a *relative to* p. *[See AUN, Exercise 10.]*

When the context is understood, we write $\mathbf{m}(a)$ and talk about the monad of a, as usual.

Theorem 1
Relative to a given context:

(1) $\mathbf{m}(0)$ *is the set of all ultrasmall numbers, together with* 0.

(2) $\mathbf{g}(0)$ *is the set of all real numbers that are not ultralarge.*

(3) $\mathbf{g}(0)$ *is a subring of* $\mathbb{R}$*; that is, if* $x, y \in \mathbf{g}(0)$, *then* $x + y \in \mathbf{g}(0)$ *and* $x \cdot y \in \mathbf{g}(0)$.

(4) $\mathbf{m}(0)$ *is an ideal of* $\mathbf{g}(0)$*; that is,* $\mathbf{m}(0) \subseteq \mathbf{g}(0)$ *and if* $x, y \in \mathbf{m}(0)$, *then* $x+y \in \mathbf{m}(0)$, *and if* $x \in \mathbf{m}(0)$ *and* $y \in \mathbf{g}(0)$, *then* $x \cdot y \in \mathbf{g}(0)$.

(5) $\mathbf{m}(a) = \{a + x : x \in \mathbf{m}(0)\}$ *and* $\mathbf{g}(a) = \{a + x : x \in \mathbf{g}(0)\}$.

Theorem 2
Let $f : [a, b] \to \mathbb{R}$ *be an internal function. Then* f *is continuous at* $x \in [a, b]$ *if and only if*

$$f[\mathbf{m}(x)] \subseteq \mathbf{m}(f(x)).$$

Theorem 3
Let $A \subseteq \mathbb{R}$ *be an internal set.*

(1) *A is open if and only if* $\mathbf{m}(a) \subseteq A$, *for each observable* $a \in A$.

(2) *A is closed if and only if* $\mathbf{m}(a) \cap A \neq \emptyset$ *implies* $a \in A$, *for each observable* $a \in A$.

The external function $\mathbf{n}_p$ is defined for $x \in \mathbf{g}_p(0)$ and has values that are observable relative to p. One might be tempted to define, say, the derivative of $\mathbf{n}_p$ using AUN, Definition 20; such an attempt would likely show that the derivative is 0 everywhere. As the function $\mathbf{n}_p$ is not constant, this would seem to contradict AUN, Theorem 40 (3). However, such an attempt would be erroneous; $\mathbf{n}_p$ is an external function, and internal methods do not apply to it. A deeper explanation is given in the next section.

2 The External Universe

In the previous section we indicated how external sets can be used for simple "bookkeeping" purposes. These external sets were just collections of internal sets. There are very useful constructions in Nonstandard Analysis that employ sets of external sets, and perhaps even more complicated objects; we give one example in the next section. For such constructions one needs more of the full power of set theory.

From now on, we prefix the word "internal" before the words "number", "set", "function", or any other concept, whenever we refer to an internal concept as defined in AUN. All names of internal sets and functions will be preceded (or sometimes succeeded, for typographical reasons) by an asterisk. Thus $^*\mathbb{N}$ is the internal set of natural numbers, $^*\mathbb{R}$ is the internal set of real numbers, $^*\sin$ is the internal sine function, and so on. One should also write $^*0, ^*1, ^*\pi$ etc. for internal numbers $0, 1, \pi$ etc., but we shall explain soon why this is not necessary.

Besides the internal objects, we will admit also objects that are not internal; we call them **external**. There is no longer any need to use exclusively boldface letters for them. The words "set" and "function" without qualification encompass henceforth both internal and external sets and functions. Thus $\mathbf{m}_p(0)$ is a set and $\mathbf{n}_p$ is a function, after all, but they are an external set and function, respectively.

As a result, we now have two domains (universes) of sets: the original *universe of internal sets*, including the standard sets and also many nonstandard objects, such as ultrasmall numbers, and the larger *universe of sets*, containing both internal and external sets. The universe of internal sets of course satisfies **ZFC** (Zermelo-Fraenkel Set Theorey with Choice). We assume that the universe of sets also satisfies the usual axioms of **ZFC**. [One of the axioms of **ZFC** is the so-called Axiom of Regularity. There are important reasons why set theorists postulate it, but it is never needed in analysis. This axiom has to fail in our universe of (internal and external) sets, so in these notes, by **ZFC** we really mean **ZFC** without the Axiom of Regularity.]

Each (traditional) mathematical concept now comes in two versions: the internal one, defined in the universe of internal sets and denoted by an asterisk, and the external one, defined in the universe of all sets, and for which we shall use the usual notation. Before attempting a more sophisticated use of external sets, we need to clarify the relationship between the two versions of each concept.

We make a basic assumption that internal sets do not acquire any new elements in the universe of all sets. Formally:

Transitivity: If X is internal and $Y \in X$, then Y is internal.

From this assumption we can deduce that many mathematical concepts (especially the elementary properties of sets and operations on them) are **absolute**: for internal sets, they give the same result whether interpreted in the internal universe or in the universe of all sets. For absolute concepts one need not worry about the distinction between the two versions, and asterisks do not have to be used. We give several examples.

Let A, B be internal sets. We have two versions of set-theoretic inclusion; the internal version:

$$A \subseteq^* B \quad \text{if} \quad x \in A \text{ implies } x \in B, \text{ for all internal } x,$$

and the external version:

$$A \subseteq B \quad \text{if} \quad x \in A \text{ implies } x \in B, \text{ for all } x.$$

If a statement is true for all x, then it is true for all internal x, so clearly if $A \subseteq B$, then $A \subseteq^* B$. Conversely, assume $A \subseteq^* B$. If $x \in A$, then x is internal by Transitivity, therefore $x \in B$. We proved that $A \subseteq B$. In conclusion, for internal sets, $A \subseteq^* B$ holds if and only if $A \subseteq B$ holds, and we can and will always write simply $A \subseteq B$. Of course, if one or both of A, B are external, then only $A \subseteq B$ is defined.

As $A =^* B$ if and only if $A \subseteq^* B$ and $B \subseteq^* A$, and similarly, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, we deduce that $A =^* B$ if and only if $A = B$, for internal sets.

Let us consider the empty set. There are two versions:

$$^*\emptyset \text{ is the internal set that has no internal elements,}$$

and

$$\emptyset \text{ is the set that has no elements.}$$

But $^*\emptyset$ has no elements as well: assume, for a contradiction, that $x \in {}^*\emptyset$, Transitivity tells us that x would have to be internal, but $^*\emptyset$ has no internal elements. We conclude that $^*\emptyset = \emptyset$.

Exercise 1

Let A, B, a, b, and f be internal. Prove that:

(1) $A \cup^* B = A \cup B$;

(2) $A \cap^* B = A \cap B$;

(3) $^*\{a, b\} = \{a, b\}$;

(4) $^*\langle a, b\rangle = \langle a, b\rangle$;

(5) f is a *-function if and only if f is a function;

(6) $^*\mathrm{dom}(f) = \mathrm{dom}(f)$.

It is a well-known and easily proved fact in set theory that all concepts defined by *restricted* statements are absolute. Restricted statements are statements where each quantifier ranges over elements of some set; that is, we can say "for all $x \in A$" and "there exists $x \in A$", but not "for all x" or "there exists x".

However, there are also important concepts whose internal and external versions differ, so the distinction has to be strictly maintained. For example, given an internal set A, we have two versions of the power set of A:

$$^*\mathcal{P}(A) = \{\text{internal } X : X \subseteq A\} \quad \text{and} \quad \mathcal{P}(A) = \{X : X \subseteq A\}.$$

It is clear that $^*\mathcal{P}(A) \subseteq \mathcal{P}(A)$. However, let us take $A = {}^*\mathbb{N}$. The set $^*\mathbb{N}$ has subsets that are not internal, for example $\{n \in {}^*\mathbb{N} : n \text{ is standard}\}$, so $\mathcal{P}(A)$ is strictly larger than $^*\mathcal{P}(A)$. Note that $^*\mathcal{P}(A)$ is internal, but $\mathcal{P}(A)$ is not (because it has elements that are not internal).

Let us now consider the two versions of natural and real numbers. Recall that the set of natural numbers was defined as the smallest inductive set, where I is *inductive* if $0 \in I$ and $x \in I$ implies $x + 1 \in I$. We will not elaborate on the fact that the notion of inductive set is absolute; that is, for internal I, the set I is *-inductive if and only if it is inductive. (For readers familiar with the usual construction of natural numbers in set theory: it is a simple consequence of $0 = \emptyset$ and $x + 1 = x \cup \{x\}$.)

But there exist inductive sets that are not internal! For example, the set $I_0 = \{x \in {}^*\mathbb{N} : x \text{ is standard}\}$ contains 0 and, whenever $n \in I_0$, $n + 1 \in I_0$, by Closure. Thus I_0 is an (external) inductive set, and therefore $\mathbb{N} \subseteq I_0 \subset {}^*\mathbb{N}$ (where $\mathbb{N}$ is the (external) set of natural numbers). In fact, we prove:

Theorem 4
The (external) natural numbers are precisely the standard natural numbers (i.e., the internal natural numbers that are always observable):

$$\mathbb{N} = \{x \in {}^*\mathbb{N} : x \textit{ is standard}\}.$$

Proof: All we need to prove is that

$$\{x \in {}^*\mathbb{N} : x \text{ is standard}\} \subseteq I, \qquad \text{for every inductive set } I.$$

So let I be inductive and assume, for a contradiction, that there is a standard $n \in {}^*\mathbb{N}$, $n \notin I$. By the Standardization Principle, there is a standard set $A \subseteq {}^*\mathbb{N}$ such that

$$k \in A \quad \text{if and only if} \quad k <^* n \text{ and } k \in I$$

holds for all standard $k \in {}^*\mathbb{N}$.

As $0 \in A$ and $A \subseteq \{k : k <^* n\}$, the set A is finite and nonempty, and so it has a greatest element $\overline{k}$. Now $\overline{k} <^* n$, $\overline{k} \in I$ and $\overline{k}$ is standard, hence $\overline{k} +^* 1 \leq^* n$, $\overline{k} +^* 1 \in I$

and $\overline{k} +^* 1$ is standard. If $\overline{k} +^* 1 <^* n$, then $\overline{k} +^* 1 \in A$, a contradiction. Therefore $\overline{k} +^* 1 = n \in I$, also a contradiction. $\square$

Examination of the set-theoretic definitions of the ordering, addition, and multiplication on natural numbers shows that, for $m, n \in \mathbb{N}$,

- $m \leq^* n$ if and only if $m \leq n$,
- $m +^* n = m + n$,
- $m \cdot^* n = m \cdot n$.

For $m, n \in \mathbb{N}$ we need not distinguish between the internal and external versions of these relations and operations. If x, y or both are in ${}^*\mathbb{N} \setminus \mathbb{N}$, then only the $*$-versions are defined. It would be logical to use only the $*$-notation; however, it is simpler to use the usual notation, without asterisks, for all internal m, n; no ambiguities can arise.

With two versions of natural numbers come two versions of induction.

Internal Induction Let $\mathcal{P}(n, a)$ be an internal statement.

- If $\mathcal{P}(0, a)$ is true;
- If $\mathcal{P}(n, a)$ implies $\mathcal{P}(n+1, a)$, for all $n \in {}^*\mathbb{N}$,

then $\mathcal{P}(n, a)$ is true for all $n \in {}^*\mathbb{N}$.

External Induction Let $\mathcal{P}(n, a)$ be any statement.

- If $\mathcal{P}(0, a)$ is true;
- If $\mathcal{P}(n, a)$ implies $\mathcal{P}(n+1, a)$, for all $n \in \mathbb{N}$,

then $\mathcal{P}(n, a)$ is true for all $n \in \mathbb{N}$.

The two distinct notions of natural numbers also lead to two distinct notions of finiteness. A set A is **finite** and has n elements ($n \in \mathbb{N}$) if there is a one-to-one sequence $\{a_i\}_{i=1}^n$ such that $A = \{a_1, \ldots, a_n\}$; we then write $|A| = n$. An internal set A is ***-finite** and has ν elements ($\nu \in {}^*\mathbb{N}$) if there is an *internal* one-to-one sequence $\{a_i\}_{i=1}^\nu$ such that $A = \{a_1, \ldots, a_\nu\}$; we then write ${}^*|A| = n$. According to these definitions, $\{1, \ldots, n\}$ for $n \in \mathbb{N}$ is finite. The set $\mathbb{N}$ itself is infinite. If $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$, then $\mathbb{N} \subseteq \{1, \ldots, \nu\}$, so $\{1, \ldots, \nu\}$ is of course infinite. This is the reason why terminology "infinitely large natural number" is sometimes used in place of "nonstandard natural number". However, $\{1, \ldots, \nu\}$ is *-finite for every $\nu \in {}^*\mathbb{N}$, whether standard or not. The set ${}^*\mathbb{N}$ is *-infinite, as well as infinite.

An important fact is that if $n \in \mathbb{N}$ and $\{a_n\}_{i=1}^n$ is a sequence where each a_i is internal, then the sequence $\{a_n\}_{i=1}^n$ itself is internal. This can be proved by external induction. As a corollary, any finite set A, all elements of which are internal, is itself internal and $|A| = {}^*|A|$. We can safely use $|A|$ as notation for the number of elements of a finite set A, whether A is internal or not.

The integers are the natural numbers and their opposites. It follows that

$$\mathbb{Z} = \{z \in {}^*\mathbb{Z} : z \text{ is standard}\}.$$

Analogously, rational numbers are the quotients of integers, so

$$\mathbb{Q} = \{q \in {}^*\mathbb{Q} : q \text{ is standard}\}.$$

The result for real numbers is a bit more involved.

In AUM, we have not given a precise definition of the set of real numbers. Instead, we observed that the set of real numbers, with the usual ordering and arithmetic operations, is an ordered field and satisfies the Completeness Axiom, and that a complete ordered field is determined uniquely, up to isomorphism. The internal set of real numbers ${}^*\mathbb{R}$ has these properties in the internal universe. In particular, every internal nonempty bounded subset of ${}^*\mathbb{R}$ has a least upper bound in the ordering $\leq^*$. However, external nonempty bounded subsets of ${}^*\mathbb{R}$ need not have a least upper bound—consider for example $\mathbb{N} \subseteq {}^*\mathbb{R}$!

Theorem 5
The (external) real numbers are precisely the standard real numbers:

$$\mathbb{R} = \{r \in {}^*\mathbb{R} : r \textit{ is standard}\}.$$

Proof: We show that $\mathbb{R}_0 = \{r \in {}^*\mathbb{R} : r \text{ is standard}\}$ is a complete ordered field in the universe of all sets. The fact that $\mathbb{R}_0$, with the ordering and arithmetic operations on ${}^*\mathbb{R}$ restricted to it, is an ordered field, follows by Closure. Henceforth, we use $\leq, +, \cdot$ in place of $\leq^*, +^*, \cdot^*$. We need to prove completeness of $\mathbb{R}_0$. Let $X \subseteq \mathbb{R}_0$ be nonempty and bounded above by $b \in \mathbb{R}_0$. By Standardization again, there is a standard set Y such that $Y \subseteq {}^*(-\infty, b]$ and $r \in Y \leftrightarrow r \in X$ holds for all $r \in \mathbb{R}$. In particular, $Y \neq \emptyset$ and $r \in Y$ implies $r \leq b$ for all $r \in {}^*\mathbb{R}_0$. It follows that Y has a least upper bound $\overline{b}$ in ${}^*\mathbb{R}$; by Closure, $\overline{b}$ is standard, so $\overline{b} \in \mathbb{R}_0$. We complete the proof by showing that $\overline{b}$ is also the least upper bound of X in $\mathbb{R}_0$. Clearly, $\overline{b}$ is an upper bound of X. Suppose $c \in \mathbb{R}_0$ is an upper bound of X in $\mathbb{R}_0$ i.e., $x \leq c$ holds for all standard $x \in Y$. Then by Closure $x \leq c$ holds for all $x \in Y$, and so $\overline{b} \leq c$. □

The set $\mathbb{R}$ is a subset of ${}^*\mathbb{R}$, and the ordering and arithmetic operations on $\mathbb{R}$ are restrictions of those on ${}^*\mathbb{R}$. In the terminology of abstract algebra, ${}^*\mathbb{R}$ is an ordered field extension of the field of real numbers $\mathbb{R}$. But it does not have the Archimedean Property: if $\nu \in {}^*\mathbb{R}$ is unlimited, then $1 + 1 + \ldots + 1 = n < \nu$ for every $n \in \mathbb{N}$. While "non-archimedean analysis" is an established field of mathematics, its results and techniques are quite different from the classical calculus of real functions. It is for this reason that we cannot expect our methods to be applicable to external functions defined on ${}^*\mathbb{R}$ or its subsets.

3 Loeb Measure

Many of the most innovative ideas in Nonstandard Analysis involve external sets. We give one example of substantial use of external sets, the celebrated Loeb measure. Our intention is only to illustrate how this construction can be carried out in our framework. The reader should consult some of the literature of Nonstandard Analysis for a deeper study of this concept and its numerous applications.

Consider the interval ${}^*[0,1]$. We fix $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $x_i = \frac{i}{N}$ and $\mathcal{T} = \{x_0, \ldots, x_{N-1}\}$. We define an internal function μ on internal subsets of ${}^*[0,1]$ as follows: if $A \subseteq {}^*[0,1]$, let

$$\mu(A) = \frac{{}^*|A \cap \mathcal{T}|}{N}.$$

Thus μ "measures the size" of the set A according to the proportion of the tags x_i that belong to A.

Theorem 6
The function μ is an internal finitely additive measure on ${}^*\mathcal{P}([0,1])$. That is,

(1) $\mu(\emptyset) = 0$ and $\mu({}^*[0,1]) = 1$;

(2) If $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Proof: Exercise. □

Exercise 2 Prove that

(1) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.

(2) If $\{A_k\}_{k=1}^n$ is an internal finite collection of mutually disjoint subsets of ${}^*[0,1]$, then

$$\mu(\bigcup_{k=1}^{n} A_n) = \sum_{k=1}^{n} \mu(A_k).$$

We now show that the internal finitely additive measure μ on ${}^*[0,1]$ gives rise to an external σ-additive measure on a certain (external) σ-algebra of subsets of ${}^*[0,1]$. We write $\mathbf{n}(x)$ in place of $\mathbf{n}_0(x)$ here and in the rest of this section.

Definition 2
For $X \subseteq {}^*[0,1]$ let

$$m^-(X) = \sup\{\mathsf{n}(\mu(A)) : A \subseteq X, A \textit{ internal}\};$$

$$m^+(X) = \inf\{\mathsf{n}(\mu(A)) : X \subseteq A, A \textit{ internal}\}.$$

The sets on the right-hand side of these definitions are (in general) external. As $\emptyset \subseteq X \subseteq {}^*[0,1]$, they are nonempty and bounded, so the supremum and infimum exist in $\mathbb{R}$ (not in ${}^*\mathbb{R}$!), and $0 \le m^-(X) \le m^+(X) \le 1$.

Definition 3
A set $A \subseteq {}^*[0,1]$ is ***Loeb measurable*** *if*

$$m^-(X) = m^+(X).$$

We let Σ^L denote the collection of all Loeb measurable sets. The function m defined for $X \in \Sigma^L$ by $m(X) = m^-(x) = m^+(X)$ is the ***Loeb measure*** *on Σ^L.*

Exercise 3 Show that $X \subseteq {}^*[0,1]$ is Loeb measurable if and only if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there are internal $A, B \subseteq {}^*[0,1]$ such that

$$A \subseteq X \subseteq B \quad \text{and} \quad \mu(B \setminus A) < \varepsilon.$$

Exercise 4 If $X \subseteq {}^*[0,1]$ is internal, then X is Loeb measurable and

$$m(X) = \mathsf{n}(\mu(X)).$$

Exercise 5 If X and Y are Loeb measurable and $X \cap Y = \emptyset$, then $X \cup Y$ is Loeb measurable and

$$m(X \cup Y) = m(X) + m(Y).$$

Exercise 6 If X is Loeb measurable, then $\complement X = {}^*[0,1] \setminus X$ is Loeb measurable and

$$m(\complement X) = 1 - m(X).$$

Exercise 7 For all $a, b \in \mathbb{R}$ such that $0 \le a < b \le 1$,

$$m(^*[a,b]) = b - a.$$

Theorem 7
The collection Σ^L is a σ-algebra containing all internal subsets of $^*[0,1]$ and all intervals $^*[a,b]$, for $a, b \in \mathbb{R}$ such that $0 \le a < b \le 1$. The Loeb measure m is a σ-additive measure on Σ^L and $m(^*[a,b]) = b - a$.

Proof: Most of the theorem follows from the results established in the exercises. It remains to prove the closure of Σ^L under countable disjoint unions and σ-additivity of m.

Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of mutually disjoint sets from Σ^L. Exercise 5 and induction show that, for every $n \in \mathbb{N}$, $\bigcup_{k=1}^{n} X_k \in \Sigma^L$ and

$$\sum_{k=1}^{n} m(X_k) = m\left(\bigcup_{k=1}^{n} X_k\right) \le m(^*[0,1]) = 1.$$

The series $\sum_{k=1}^{\infty} m(X_k)$ therefore converges.

Let $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. We fix $K \in \mathbb{N}$ such that

$$\sum_{k=K}^{\infty} m(X_k) < \frac{\varepsilon}{2}.$$

Since $X_n \in \Sigma^L$, there exist internal sets A, B with $A \subseteq X_n \subseteq B$ and $\mu(B \setminus A) < \frac{\varepsilon}{2^{n+1}}$. By the Principle of Limited Choice (see the next section), there are internal sequences $\{A_n\}_{n \in {^*\mathbf{N}}}$ and $\{B_n\}_{n \in {^*\mathbf{N}}}$ such that, for all $n \in \mathbb{N}$,

$$A_n \subseteq X_n \subseteq B_n \quad \text{and} \quad \mu(B_n \setminus A_n) < \frac{\varepsilon}{2^{n+1}}.$$

Now $\sum_{k=K}^{n} \mu(A_k) \simeq_1 \sum_{k=K}^{n} m(A_k) \le \sum_{k=K}^{n} m(X_k) < \frac{\varepsilon}{2}$, so $\sum_{k=K}^{n} \mu(A_k) < \frac{\varepsilon}{2}$. It follows that there is $\nu \in {^*\mathbb{N}} \setminus \mathbb{N}$ such that, for all $n \le \nu$,

$$A_n \subseteq B_n, \quad \mu(B_n \setminus A_n) < \frac{\varepsilon}{2^{n+1}}, \quad \text{and} \quad \sum_{k=K}^{n} \mu(A_k) < \frac{\varepsilon}{2} \tag{*}$$

(If (*) fails for some ν, then there is a least ν where it fails, but this ν has to be nonstandard. Then (*) holds for $\nu - 1$, which is still nonstandard.)

We now let $A = \bigcup_{k=1}^{K-1} A_k$ and $B = \bigcup_{k=1}^{\nu} B_k$; also let $X = \bigcup_{k=1}^{\infty} X_k$. The sets A and B are internal, $A \subseteq X \subseteq B$, and

$$\begin{aligned}\mu(B \setminus A) = \mu(B) - \mu(A)& \\ &= \sum_{k=1}^{\nu} \mu(B_k) - \sum_{k=1}^{K-1} \mu(A_k) \\ &= \sum_{k=1}^{\nu} (\mu(B_k) - \mu(A_k)) + \sum_{k=K}^{\nu} \mu(A_k) \\ &< \sum_{k=1}^{\nu} \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

So $X \in \Sigma^L$ and m is σ-additive. $\square$

We conclude this section by showing how Lebesgue measure on $[0,1]$ can be obtained from Loeb measure.

Definition 4
*A set $S \subseteq [0,1]$ is **Lebesgue measurable** if*

$$\mathbf{n}^{-1}[S] = \{x \in {}^*[0,1] : \mathbf{n}(x) \in S\}$$

is Loeb measurable. Σ is the collection of Lebesgue measurable sets. For $S \in \Sigma$ we let $\ell(S) = m(\mathbf{n}^{-1}[S])$.

Exercise 8 Σ is a σ-algebra and $[a,b] \in \Sigma$ for all $a, b \in \mathbb{R}$, $0 \leq a < b \leq 1$.

Exercise 9 ℓ is a σ-additive measure on Σ and $\ell([a,b]) = b - a$.

Theorem 8
For $S \in \Sigma$, we have

$$\ell(S) = \inf\{\ \ell(O \cap [a,b]) : S \subseteq O,\ O \textit{ open}\}.$$

It follows from these exercises and theorems that ℓ coincides with the Lebesgue measure as traditionally defined.

We note that Σ and ℓ are external; ℓ is the Lebesgue measure in the universe of all sets. But what about the σ-algebra ${}^*\mathcal{M}({}^*[0,1])$ of Lebesgue measurable sets and the Lebesgue measure ${}^*m$ that we constructed in the internal universe in the notes on Generalized Riemann integral? We already pointed out that for each object defined in **ZFC** our theory has an external version and an internal version, marked by an asterisk. In the following section we show that * maps a fragment of the external universe "isomorphically" into the internal universe. In particular, ${}^*\Sigma$ and ${}^*\ell$ are defined, and they are precisely the internal σ-algebra of Lebesgue measurable sets ${}^*\mathcal{M}({}^*[0,1])$ and the internal Lebesgue measure ${}^*m$, respectively.

4 Relative Set Theory Revisited

First, we summarize the axioms of the theory of internal and external sets outlined in this chapter. The language of discourse needs, in addition to $\in$ and $\sqsubseteq$, a primitive unary predicate symbol $\mathbf{I}$, with $\mathbf{I}(x)$ read "x is internal." Informally, we write $x \in \mathbf{I}$ in place of $\mathbf{I}(x)$. We postulate all the axioms of **ZFC** except Regularity, with the understanding that the axiom schemata of Separation and Replacement apply to arbitrary statements in the $\in$-$\sqsubseteq$-$\mathbf{I}$-language. If $\mathcal{P}$ is a statement in the $\in$-$\sqsubseteq$-language, ${}^*\mathcal{P}$ is obtained from $\mathcal{P}$ by restricting all quantifiers to internal sets, that is, by replacing each occurence of $\forall x$ and $\exists x$ in $\mathcal{P}$ by $\forall x \in \mathbf{I}$ and $\exists x \in \mathbf{I}$, respectively. In informal usage, the statement $\mathcal{P}$ may contain other previously defined notions such as $\subseteq, \emptyset, \mathbb{N}$ and so on; it is understood that such notions are replaced in ${}^*\mathcal{P}$ by the corresponding internal notions, such as $\subseteq^*, {}^*\emptyset, {}^*\mathbb{N}$. We further postulate that ${}^*\mathcal{P}$ is an axiom, for every axiom $\mathcal{P}$ of **RBST**. Informally, the internal sets, with the relations $\in$ and $\sqsubseteq$, satisfy **RBST**.

The relation $\sqsubseteq$ applies only to internal sets, and the internal universe is transitive, so we also postulate

$$(\forall x, y)(x \sqsubseteq y \rightarrow \mathbf{I}(x) \wedge \mathbf{I}(y))$$

and

$$(\forall x, y)\ (\mathbf{I}(x) \wedge y \in x \rightarrow \mathbf{I}(y)).$$

Finally, some of our principles about internal sets need to be strengthened to allow external parameters. We recall that $\mathbf{S}(x)$ stands for $x \sqsubseteq 0$ (x *is standard*).

Full Standardization for the standard universe:

Let $\mathcal{P}(z, x_1, \ldots, x_k)$ be any statement in the $\in$-$\sqsubseteq$-$\mathbf{I}$-language. For all $x_1, \ldots, x_k$ (even external),

$$(\forall^{\mathbf{I}} A)(\exists^{\mathbf{S}} B)(\forall^{\mathbf{S}} z)(z \in B \leftrightarrow z \in A \wedge \mathcal{P}(z, x_1, \ldots, x_k)).$$

Full Limited Choice for the standard universe:

Let $\mathcal{P}(x, y, x_1, \ldots, x_k)$ be any statement in the $\in$-$\sqsubseteq$-$\mathbf{I}$-language, and let A be internal. If

$$(\forall^{\mathbf{S}} x \in A)(\exists y \in \mathbf{I})\mathcal{P}(x, y, x_1, \ldots, x_k),$$

then there is an internal function F with domain A such that

$$(\forall^{\mathbf{S}} x \in A)\mathcal{P}(x, F(x), x_1, \ldots, x_k).$$

Full Standardization was used in the proof of Theorem 4 (the external parameter is I), and Theorem 5 (the external parameter is X). Full Limited Choice was used in the proof of Theorem 7 (the external parameter is the sequence $\{X_k\}_{k=1}^{\infty}$).

The list of axioms given here is not complete in any sense. However, it is sufficient to obtain, in our setting, something like the superstructure framework of Nonstandard Analysis.

Full Standardization implies that, for every set $A \subseteq \mathbb{R}$, there is a unique internal set ${}^*A \subseteq {}^*\mathbb{R}$, ${}^*A \in \mathbf{S}$, such that

$$(\forall x \in \mathbb{R})(x \in {}^*A \leftrightarrow x \in A).$$

So $*$ maps $\mathcal{P}(\mathbb{R})$ into ${}^*\mathcal{P}({}^*\mathbb{R}) \cap \mathbf{S}$. Conversely, if $B \subseteq {}^*\mathbb{R}$ and $B \in \mathbf{S}$, let $A = B \cap \mathbb{R}$; it then follows that $B = {}^*A$. We conclude that $*$ is a one-to-one correspondence between $\mathcal{P}(\mathbb{R})$ and ${}^*\mathcal{P}({}^*\mathbb{R}) \cap \mathbf{S}$. The same argument can be used to extend $*$ to a one-to-one correspondence between $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ and ${}^*\mathcal{P}({}^*\mathcal{P}({}^*\mathbb{R})) \cap \mathbf{S}$, and so on by induction. In fact, it can be proved from our axioms that there is a set $\mathbf{U}$ such that

(1) $\mathbb{R} \subseteq \mathbf{U}$ and $\mathbb{R} \in \mathbf{U}$;

(2) If $X \in \mathbf{U}$, then $\mathcal{P}(X) \in \mathbf{U}$; and

(3) There is a one-to-one mapping $*$ of $\mathbf{U}$ onto $\mathbf{S}$ such that $X \in Y \leftrightarrow {}^*X \in {}^*Y$ holds for all $X, Y \in \mathbf{U}$.

It follows in particular that $\mathbf{S}$ is a set. Thence also $\mathbf{I}$ is a set—the set of all internal sets ($\mathbf{I}$ is the union of (all elements of) the set $\mathbf{S}$, by Boundedness). This does not create Russell's paradox, because $\mathbf{S}$ and $\mathbf{I}$ are not internal sets, of course. They are external, just like $\mathbb{R}$ and $\mathbb{N}$. The present setup resembles the so-called **superstructure framework** widely used in Nonstandard Analysis. Sets of the form ${}^*A$ with $A \in \mathbf{U}$ are called **standard**; they are precisely the sets in $\mathbf{S}$. The sets in $\mathbf{I}$ that are not in $\mathbf{S}$ are called **nonstandard**. The principle that nonstandard analysts call Transfer is satisfied: If $\mathcal{P}$ is a statement in the $\in$-language where all quantifiers are restricted (i.e., of the form $\forall x \in y$ or $\exists x \in y$), then

$$(\forall A_1, \ldots, A_k \in \mathbf{U})\ (\mathcal{P}(A_1, \ldots, A_k) \leftrightarrow {}^*\mathcal{P}({}^*A_1, \ldots, {}^*A_k).$$

We state without proof that the axiomatic system described in this section is consistent (provided ZFC is); for a proof of its consistency and more about external sets in relative set theory, see
KH, *Relative set theory: Some external issues*, Journ. Logic and Analysis 2:8, 2010, 1 - 37.
One can carry out most of the nonstandard arguments (for example, all of those in Goldblatt's book) in this setting. In addition, our superstructure $\mathbf{U}$ satisfies all of **ZFC**, and our universe of internal sets $\mathbf{I}$ is stratified into levels by $\sqsubseteq$; these features allow for constructions that are not immediately possible in superstructures of Nonstandard Analysis.

The reader is likely to have noticed the foundational shift that gradually took place in these notes. Throughout AUN, we maintained that standard sets, viewed as having also ideal, internal elements, are the "real" sets, the sets of traditional mathematics. Here we paint a picture of a much larger universe than the internal one, a universe with its own versions of fundamental mathematical objects, such as the set of natural numbers $\mathbb{N}$ and the set of real numbers $\mathbb{R}$. On this view it seems natural to conclude that these external sets are the "real" sets, and in particular, that $\mathbb{N}$ and $\mathbb{R}$ are the "real" natural and real numbers, respectively, while ${}^*\mathbb{N}$ and ${}^*\mathbb{R}$ are merely some helpful, but somewhat peculiar, non-archimedean extensions of $\mathbb{N}$ and $\mathbb{R}$. This is in fact the position of Nonstandard Analysis, and the terminology used in Sections 2 – 4 is chosen to correspond to it (here we use "natural numbers" for the elements of $\mathbb{N}$, and "internal natural numbers" for the elements of ${}^*\mathbb{N}$, and so on).

We conclude this discussion with two points. First, we would like to stress that the issue whether the standard or the external sets are the "real" sets is not a mathematical question, but a philosophical one. The answer does not effect the validity of the mathematics. Second, both positions are defensible. We managed to develop a large portion of classical analysis using infinitesimal methods, while working only with internal sets. Nonstandard Analysis accomplishes the same goal, but with substantially greater dose of technicalities, in our opinion.

5 About fine levels

In this section we work in **RBST** and take a more general look at the theory of a-accessibility used in the notes on Generalized Riemann Integral (GRI).

We fix a context p, an arbitrary set a, and a set A observable relative to p such that $a \in A$ (existence of such A is guaranteed by Boundedness). We let $\mathbf{S}_p = \{x : x \sqsubseteq p\}$.

Definition 5
*A set x is a-**accessible** relative to p if $x = f(a)$, for some function f defined on A and observable relative to p.*

If a and x are real numbers and $A = \mathbb{R}$, this is just the definition from Section 1 of GRI; but now a and x can be arbitrary sets. Accordingly, the function F can have arbitrary values; it need not be a real-valued function.

We write $x \in \mathbf{S}_p[a]$ to denote that x is a-accessible relative to $\mathbf{S}_p$. The arguments in Section 1 of GRI generalize directly to show that, for $\mathbf{S}_p \subseteq \mathbf{S}_q$ and $a \in \mathbf{S}_q$ we have

$$\mathbf{S}_p \subseteq \mathbf{S}_p[a] \subseteq \mathbf{S}_q.$$

We also have

Theorem 9 (a-closure principle)
Given a statement $\mathcal{P}(y, a, \alpha)$ in the $\in$-language: If there exists a set y for which the statement is true, then there exists a set y in $\mathbf{S}_p[a]$ for which the statement is true.

The proof is a minor modification of the special case, Theorem 3 in GRI.

We can think of the $\mathbf{S}_p[a]$'s as a refinement of the stratification of the universe of sets into "levels of observability" $\mathbf{S}_p$. These "fine levels" unfortunately do not have the uniform properties provided by Stability, and they fail to satisfy Standardization and Bounded Idealization as well. They are not even linearly ordered: it is easy to show that there are a, b such that

$$b \notin \mathbf{S}_p[a] \quad \text{and} \quad a \notin \mathbf{S}_p[b].$$

In fact, their properties depend essentially on the choice of a. This is why we find the "coarse levels" used in AUN much more suitable for general-purpose analysis. However, for special purposes the fine levels may sometimes be convenient, as GRI demonstrates.

Here we examine the structure of the $\mathbf{S}_p[a]$'s in some more detail, in order to relate them to constructions familiar from Nonstandard Analysis.

Let $A \in \mathbf{S}_p$ and pick $a \in A$. We consider the set

$$V_a = \{X \subseteq A : a \in X\}.$$

By Standardization, this set has a shadow in $\mathbf{S}_p$; that is, there is a uniquely determined set $U_{p,a}$ in $\mathbf{S}_p$ such that, for all X in $\mathbf{S}_p$ with $X \subseteq A$ we have

$$X \in U_{p,a} \quad \text{if and only if} \quad a \in X.$$

Exercise 10 (Answer page 22)
Show that the collection $U = U_{p,a}$ has the following properties:

(1) $A \in U$, $\emptyset \notin U$.

(2) If $X \in U$ and $X \subseteq Y \subseteq A$, then $Y \in U$.

(3) If $X, Y \in U$, then $X \cap Y \in U$.

(4) If $X \cup Y \in U$, then either $X \in U$ or $Y \in U$.

[Hint: prove (1)-(4) first under the assumption that X, Y are in $\mathbf{S}_p$; then use the fact that U is in $\mathbf{S}_p$ and Closure.]

Definition 6
*A collection U of subsets of A with properties (1)-(4) is called an **ultrafilter** over A.*

Exercise 11 (Answer page 19)
Let $\mathcal{P}(A)$ be the power set of A, i.e., the collection of all subsets of A. Given an ultrafilter U over A, define the function $\mu : \mathcal{P}(A) \to \{0, 1\}$ by:

$$\mu(X) = \begin{cases} 1 & \text{if } X \in U; \\ 0 & \text{otherwise.} \end{cases}$$

The function μ is the characteristic function of U.
Prove that μ is a finitely additive measure on the σ-algebra $\mathcal{P}(A)$.
Conversely, if μ is a finitely additive measure on $\mathcal{P}(A)$ with range $\{0, 1\}$, then $U = \{X \subseteq A : \mu(X) = 1\}$ is an ultrafilter over A.

Another easy exercise is the following.

Exercise 12 (Answer page 22)
Show that

(1) If a is in $\mathbf{S}_p$, then $\{a\} \in U_{p,a}$ and $U_{p,a} = V_a$.

(2) If a is not in $\mathbf{S}_p$, then no finite set belongs to $U_{p,a}$.

Ultrafilters that contain no finite sets are called **free**; hence $U_{p,a}$ is a free ultrafilter if and only if a is not in $\mathbf{S}_p$.

By our construction, each a determines an ultrafilter $U_{p,a}$ in $\mathbf{S}_p$. Conversely, every ultrafilter in $\mathbf{S}_p$ is $U_{p,a}$ for a suitable a.

Exercise 13 (Answer page 22)
Let U be an ultrafilter in $\mathbf{S}_p$. Show that $U = U_{p,a}$ for some a.
[Hint: Using Bounded Idealization, show that there exists a such that $a \in X$ for all $X \in U$ which are in $\mathbf{S}_p$.]

We write $F \in \mathbf{S}_p^A$ to indicate that F is a function in $\mathbf{S}_p$ and defined on A.

Let U be an ultrafilter in $\mathbf{S}_p$. For $F, G \in \mathbf{S}_p^A$ we define:

$$
\begin{aligned}
F =_U G &\quad \text{if and only if} \quad \{x \in A : F(x) = G(x)\} \in U; \\
F \in_U G &\quad \text{if and only if} \quad \{x \in A : F(x) \in G(x)\} \in U.
\end{aligned}
$$

Exercise 14 (Answer page 23)
Prove the following statements.

(1) $F =_U F$;

(2) If $F =_U G$, then $G =_U F$;

(3) If $F =_U G$ and $G =_U H$, then $F =_U H$;

(4) If $F =_U H$ and $F \in_U G$, then $H \in_U G$;

(5) If $G =_U H$ and $F \in_U G$, then $F \in_U H$.

The structure $\mathbf{S}_p^A$ with the relations $=_U$ and $\in_U$ is called the **ultrapower** of $\mathbf{S}_p$ modulo the ultrafilter U.

Exercise 15 (Answer page 23)
Let $U = U_{p,a}$. Prove the following statements.

(1) $F =_U G$ if and only if $F(a) = G(a)$.

(2) $F \in_U G$ if and only if $F(a) \in G(a)$.

Put into words, this exercise shows that the correspondence

$$F \mapsto F(a)$$

is an isomorphism between the ultrapower of $\mathbf{S}_p$ modulo $U_{p,a}$ and the fine level $\mathbf{S}_p[a]$. The mathematicians who practice Nonstandard Analysis often work in the framework of ultrapowers of a fragment of the universe of sets. Occasionally, the choice of the ultrafilter is important. According to the previous discussion, use of ultrapowers amounts, in our framework, to work with fine levels.

Answers to the exercises

Answer to Exercise 10, page 19

The context level is $\mathbf{S}_p$. For (1) $A \in U$ is clear since $A \in \mathbf{S}_p$ and $a \in A$. Also $a \notin \emptyset$ so $\emptyset \notin U$.

For (2)–(4), it is enough to prove it for X, Y in the context level and then use Closure. For (2) if $X \in U$ then $a \in X$ and if $X \subseteq Y \subseteq A$, then also $a \in Y$ i.e., $Y \in U$. For (3) if $X, Y \in U$ then $a \in X$ and $a \in Y$ so $a \in X \cap Y \subseteq A$ so $X \cap Y \in U$. For (4) if $X \cup Y \in U$ then $a \in X \cup Y$ so $a \in X$ or $a \in Y$, but $X, Y \subseteq X \cup Y \subseteq A$ i.e., $X \in U$ or $Y \in U$.

Answer to Exercise 11, page 19

$\Longrightarrow$: Suppose that U is an ultrafilter and let μ be as given. We check the properties (1)-(3) of the definition of additive measure. (1) holds since $\mu(X) \geq 0$, for all $X \in U$. For (2), since $A \in U$ then $\mu(A) = 1 > 0$, and since $\emptyset \notin U$ then $\mu(\emptyset) = 0$. Finally, for (3) let X and Y be disjoint. If neither belong to U then $X \cup Y \notin U$ so $\mu(X \cup Y) = 0 = 0 + 0 = \mu(X) + \mu(Y)$. If X belongs to U then $A \setminus X \notin U$ and since X, Y are disjoint $Y \subseteq A \setminus X$, so $Y \notin U$. But $X \subseteq X \cup Y$, so $X \cup Y \in U$. This implies that $\mu(X \cup Y) = 1 = 1 + 0 = \mu(X) + \mu(Y)$. The case when Y belongs to U is identical. This shows that μ is finitely additive.

$\Longleftarrow$: Suppose that μ is a finitely additive and let U be as given. We check the properties (1) - (4) of the definition of ultrafilter. (1) holds since $\mu(A) = 1$ so $A \in U$, and $\mu(\emptyset) = 0$ so $\emptyset \notin U$. For (2), if $X \in U$ and $X \subseteq Y$ then $\mu(Y) \geq \mu(X) = 1$, so $\mu(Y) = 1$ i.e., $Y \in U$. For (3), suppose that $\mu(X) = \mu(Y) = 1$. Hence, $\mu(A \setminus X) = 0$, since $1 = \mu(A) = \mu(X \cup (A \setminus X)) = \mu(X) + \mu(A \setminus X)$. Then $\mu(Y \setminus X) = 0$ since $Y \setminus X \subseteq A \setminus X$. But we also have $Y = (X \cap Y) \cup (Y \setminus X)$, so $1 = \mu(X \cap Y) + 0$, so $\mu(X \cap Y) = 1$ i.e., $Y \in U$. For (4), suppose that $\mu(X \cup Y) = 1$. If both $\mu(X) = \mu(Y) = 0$ then $\mu(X \cup Y) = \mu(X) + \mu(Y \setminus X) = 0$, since $\mu(Y \setminus X) \leq \mu(Y) = 0$, a contradiction.

Answer to Exercise 12, page 19

(1) If $a \in \mathbf{S}_p$ then $\{a\} \in \mathbf{S}_p$ by Closure and $a \in \{a\} \subseteq A$ so $\{a\} \in U$. If fact, if $a \in \mathbf{S}_p$ then V_a (which is entirely defined with a and A) is also in $\mathbf{S}_p$, so $V_a = U_{p,a}$.

(2) If there is $F \in U$ finite, then by Closure, there is a finite $F \in U$ in $\mathbf{S}_p$. But $F \in \mathbf{S}_p$ is in U if and only if $a \in F$. Since F is finite, all its elements are also in $\mathbf{S}_p$, so $a \in \mathbf{S}_p$.

Answer to Exercise 13, page 20

Let $U \in \mathbf{S}_p$ be an ultrafilter on A. We first show that there exists $a \in A$ such that $a \in X$, for all $X \in U$. The argument goes like the proof of Saturation (see the Appendix in AUM; in fact, it is a consequence of Saturation).

Let $U' = \{X_1, \dots, X_n\}$ be a finite subcollection of U in $\mathbf{S}_p$. Since U is an ultrafilter, $X_1 \cap \dots \cap X_n \neq \emptyset$. Thus there exists $a' \in \cap U'$. By Bounded Idealization, we conclude that there exists $a \in A$ such that $a \in X$ for all $X \in U$ in $\mathbf{S}_p$.

Now let $Y \in \mathbf{S}_p$ such that $Y \notin U$. We show that $a \notin Y$. Since U is an ultrafilter and $A \setminus Y \in U$ (as $A = Y \cup (A \setminus Y)$) and $A \in U$), then $a \in A \setminus Y$. This shows that $a \notin Y$.

In all, we have shown that for $X \in \mathbf{S}_p$, we have $X \in U$ if and only if $a \in X$. This implies that $U = U_{p,a}$.

Answer to Exercise 14, page 20

(1) and (2) are clear. (3) is clear since

$$\{x \in A : F(x) = G(x)\} \cap \{x \in A : G(x) = H(x)\} \subseteq \{x \in A : F(x) = H(x)\}.$$

The sets on the left belong to U by definition, so the intersection is also in U. The set on the right contain a set in U so it is in U.

The argument for (4) and (5) are similar with an instance of $\in$ replacing $=$.

Answer to Exercise 15, page 20

This is a simple translation: $F =_U G$ if and only if $\{x \in A : F(x) = G(x)\} \in U_{a,A}$ if and only if $a \in \{x \in A : F(x) = G(x)\}$ ($F, G \in \mathbf{S}_p$, so this set is necessarily in $\mathbf{S}_p$) if and only if $F(a) = G(a)$.

The second claim is proved similarly.